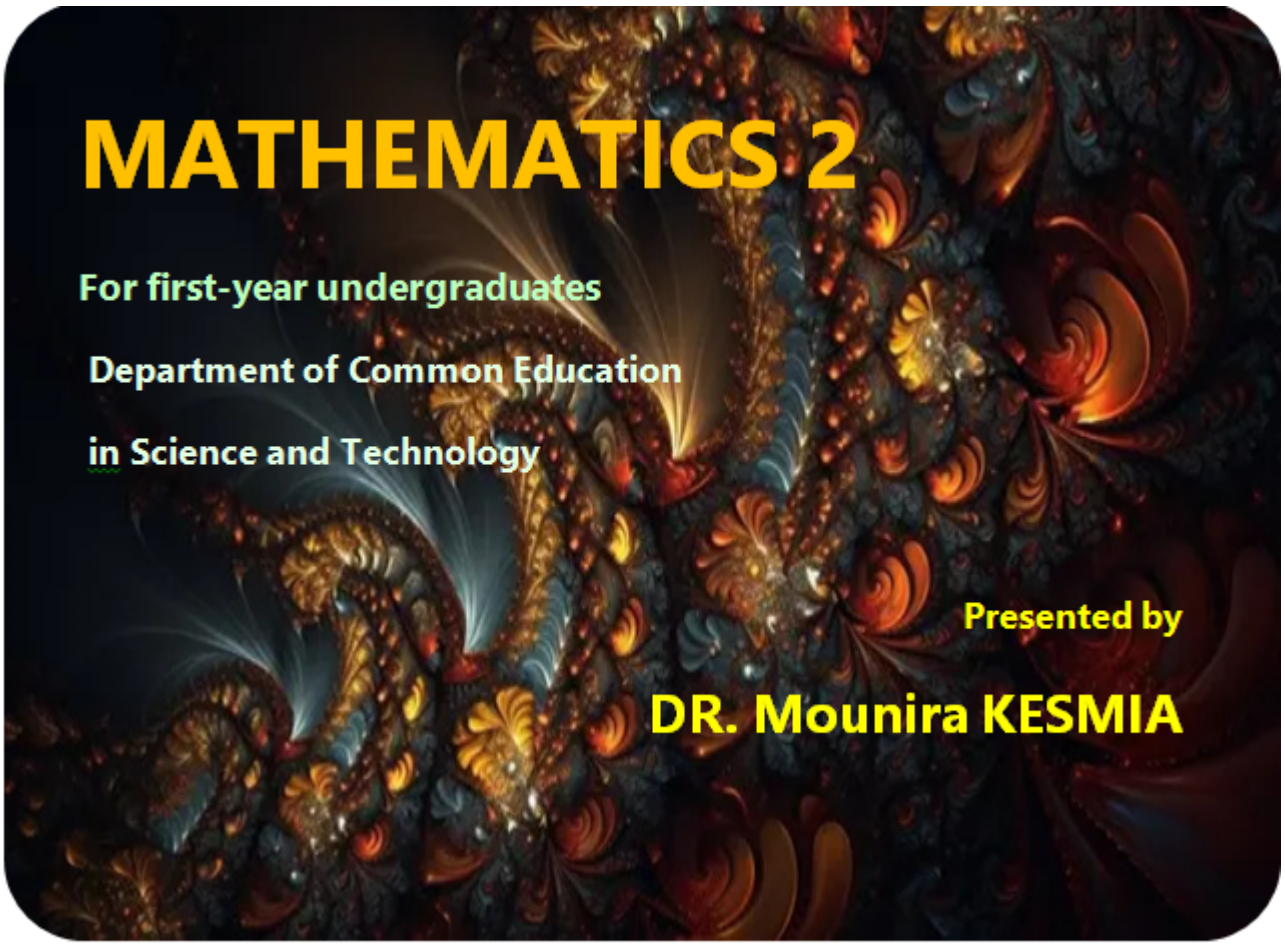
People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research Research

University Constantine 1, Frères Mentouri

Faculty of Exact Sciences

Department of Mathematics



"Second Semester of the Academic Year 2023-2024"

This Handout is dedicated to my husband and daughters for their support and understanding

ACKNOWLEDGMENTS

In the foremost position, I extend heartfelt gratitude to 'ALLAH,' whose give me strength and patience enabled me to accomplish this Handout, despite encountering various challenges

And in the second position I would like to sincerely thank the referees of this handout for their valuable time and expertise

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Introduction

Welcome to the Mathematics 2 handout, tailored for first-year science and technology students in Algerian universities.

This handout serves as a guide through topics such as vector spaces, matrices, linear systems, integrals, and multivariable functions. Our goal is to foster a solid understanding of these fundamental mathematical concepts while enhancing English Language skills. This handout is organized into six pivotal chapters, meticulously aligned with the Mathematics 2 syllabus. Here are some brief descriptions for all chapters:

Chapter 1: Vector Spaces and Linear Maps

In mathematics, a vector space involves vectors undergoing operations like addition and scalar multiplication. A basis, a set of linearly independent vectors spanning the space, uniquely represents vectors. The dimension is the basis's vector count. In linear maps, terms like kernel (set of vectors mapped to zero) and image (set of produced vectors) are key. Linear maps can be bijective (injective and surjective), injective (one-to-one), or surjective (onto). These concepts are vital for understanding mathematical structures with applications in physics, computer science, and engineering.

Chapter 2: Matrices and Determinants

Building upon the foundation laid in the first chapter, we transition seamlessly to the realm of matrices and determinants. Matrices, comprised of rows and columns, serve as versatile structures for organizing and manipulating data across various disciplines. They

provide a concise representation of complex relationships, making them indispensable in physics, computer science, and engineering. Determinants, numerical values associated with square matrices, play a pivotal role in matrix analysis. They determine matrix invertibility and help understand scaling factors in linear transformations. Matrices and determinants find extensive utility in solving equations, analyzing transformations, and addressing challenges in optimization and statistics, making them indispensable tools in diverse scientific and technological applications.

Chapter 3: Solving Linear Systems

Solving linear systems involves finding the values of variables that satisfy a set of linear equations. These equations describe relationships between variables in a straightforward, linear fashion. Various methods, such as substitution, elimination, and matrix techniques, are employed to determine the unique solution if it exists or to identify cases where the system has no solution or an infinite number of solutions. Solving linear systems is a foundational skill in mathematics with applications spanning physics, engineering, economics, and other fields.

Chapter 4: Primitives and Integrals

Integration is a fundamental concept in calculus that involves finding the accumulation of quantities, typically represented as the area under a curve on a graph. It provides a way to calculate the total or accumulated effect of a varying quantity over a given interval. The definite integral yields a specific numerical value, representing the net change or total quantity, while the indefinite integral provides a family of functions that, when differentiated, yield the original integrrand. Integration finds widespread applications in physics, engineering, economics, and various scientific disciplines, allowing for the analysis of rates of change and total effects in diverse scenarios.

Chapter 5: Differential Equations

As we progress, we encounter the fascinating realm of differential equations. This chapter unravels the language of change, exploring differential equations as a powerful tool in modeling dynamic phenomena across various scientific and technological disciplines.

Chapter 6: Functions of Several Variables

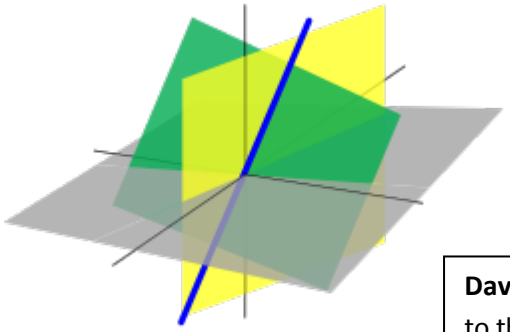
Our exploration concludes with a multifaceted study of functions in multivariables. We investigate the extension of single-variable concepts to higher dimensions, paving the way for a more nuanced understanding of mathematical structures and relationships.

Within these chapters, students and professors will find a rich tapestry of mathematical knowledge interwoven with linguistic proficiency, bridging theory and application. Mathematics 2 is not merely a guide but a key to unlocking the potential within each mathematical concept, fostering a holistic approach to learning that transcends disciplinary boundaries.

As you navigate through these chapters, may the language of mathematics become a fluent medium for expressing the beauty and precision inherent in the world of science and technology.

Vector Space and Linear Map

Many mathematicians have contributed to **vector space theory and linear maps**. Although it's hard to attribute the founding to a few, notable contributors include:



Hermann Grassmann (1809–1877): Grassmann introduced the concept of vectors and vector spaces in his work "Die lineale Ausdehnung Lehre" (1844), laying the groundwork for modern vector space

David Hilbert (1862–1943): Hilbert contributed substantially to the abstract formulation of vector spaces and linear algebra. His work on axiomatic systems and invariant theory greatly influenced the development of these theories.

Giuseppe Peano (1858–1932): Peano contributed to the early development of vector spaces and linear algebra, particularly through his work on functional analysis and the foundations of mathematics.

Felix Klein (1849–1925): Klein played a crucial role in the abstract development of geometry and algebra, which provided the framework for understanding vector spaces and linear transformations.

Chapter 1

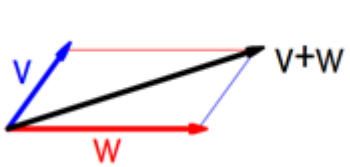
Vector Spaces and Linear Maps

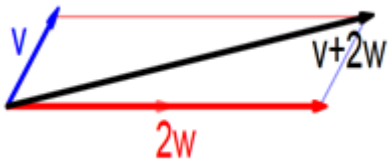
1.1 Vector Space

A **Vector space theory** is concerned with two different kinds of mathematical objects, called **'vectors'** and **'scalar'**.

A **vector space** (also called **linear space**) is set whose elements, often called **'vectors'**, may be added together and multiplied by numbers called **'scalars'**.

The **vector addition** and **scalar multiplication** must satisfy certain properties.





In fact, the **scalars** must form what mathematicians call a **'field'**. This means, a set whose elements, may be added, multiplied, subtracted, divided together. Examples of fields include the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$. However, $\mathbb{N}$ is not a field (we cannot subtract $(-)$ or divide($\div$)) and $\mathbb{Z}$ is not a field (we cannot divide ($\div$)).

Example: (Application of space in physics)

Vectors can be used to represent physical quantities: displacement velocity, acceleration, electromagnetic force, gravitational force, ...etc

1.1.1 Definitions and Basic Properties

A **vector space over the field** $\mathbb{K}$ is a nonempty set V of elements, called **vectors**, equipped with two operations:

Internal operation External operation

1. **An internal operation (vector addition)**

$$\begin{array}{rcl} + : V \times V & \rightarrow & V \\ (u, v) & \mapsto & u + v \end{array}$$

V **is closed** under **addition** $+ : u + v \in V$

2. **An external operation (scalar multiplication)**

$$\begin{array}{rcl} . : \mathbb{K} \times V & \rightarrow & V \\ (\alpha, u) & \mapsto & \alpha.u \end{array}$$

V **is closed** under **scalar multiplication** $(.) : \alpha.u \in V$

For having a vector space, the eight following properties must be satisfied

$\forall u, v, w \in V$ and $\alpha, \beta \in \mathbb{K}$

- **Vector addition property**

  1. Commutativity $u + v = v + u$
  2. Associativity $(u + v) + w = u + (v + w)$
  3. Identity element is the **zero vector** 0_V $\exists\, 0_V \in V : v + 0_V = v$

4. Inverse element $\exists\ (-v) \in V : v + (-v) = 0_V$

- **Scalar multiplication property**

1. Compatibility of **scalar multiplication** with **ordinary multiplication**

$$\alpha.(\beta.v) = (\alpha\beta).v$$

2. Identity element of the scalar multiplication is 1

$$1.v = v$$

3. Distributivity of **scalar multiplication** with respect to **vector addition**

$$\alpha.(u + v) = \alpha.u + \alpha.v$$

4. Distributivity of **scalar multiplication** with respect to **scalar addition**

$$(a + b).v = a.v + b.v$$

Notice:

- The vector space is called vector space over the field $\mathbb{K}$ or $\mathbb{K}$-vector space.
- The term **real vector space** is used to specify the nature of scalars: **real coordinate space.**

Examples:

1. The plan $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ is a real vector space of all two ordered real numbers, (x, y) is a vector whose components the two scalars x and y.
2. $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ is a real vector space of all three ordered real numbers, (x, y, z) is vector whose components the three scalars x, y and z.
3. $\mathbb{R}^n = \{X = (x_1, x_2..., x_n) : x_i \in \mathbb{R}, i = \overline{1, n}\}$ is a set of all n ordered real numbers called n-Tuples. $(x_1, x_2..., x_n)$ is a vector whose components the n scalars $x_i \in \mathbb{R}$.

The set $\mathbb{R}^n$ equipped with the vector addition and multiplication is a vector space

$$X + Y = (x_1, x_2..., x_n) + (y_1, y_2..., y_n) = (x_1 + y_1, x_2 + y_2 + ..., x_n + y_n) \in \mathbb{R}^n$$

$$\alpha.X = \alpha(x_1, x_2..., x_n) = (\alpha x_1, \alpha x_2...\alpha x_n) \in \mathbb{R}^n$$

- $(0, 0..., 0)$ is **the zero vector** of $\mathbb{R}^n$ and denoted $\mathbf{0}$.
- $-X$ **is the negative of** X**.** It a vector having the same length (or magnitude) as X**, but lies in the opposite direction to** X**.**
- $X = (x_1, x_2..., x_n)$ is called **row vector.**
- $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is called **column vector.**

The set $\mathbb{R}^n$ equipped with addition and scalar multiplication verifies all the properties of the vector space.

4. $F = \{f \setminus f : \mathbb{R} \to \mathbb{R}\}$ the set of all real functions is a vector space over $\mathbb{R}$. Then the function is a vector.

   Let $f, g \in F, \alpha \in \mathbb{R}:$ $f+g$ is defined by $(f+g)(x) = f(x)+g(x) \implies f+g : \mathbb{R} \to \mathbb{R}$

   $f + g \in F$. Thus, F is **closed under addition.**

   $\alpha.f$ is defined by $(\alpha.f)(x) = \alpha f(x)$

   $\implies \alpha.f : \mathbb{R} \to \mathbb{R}$

   $\implies \alpha.f \in F$. Thus, F is **closed under scalar multiplication.**

   The set F equipped with addition and multiplication verifies all the properties of the vector space.

5. Consider $\mathcal{P}_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 \setminus a_0, a_1, a_2, a_3 \in \mathbb{R}\}$. The set of polynomials of degree three or less. It is vector space under the operation:

$$(a_0+a_1x+a_2x^2+a_3x^3)+(b_0+b_1x+b_2x^2+b_3x^3) = (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+(a_3+b_3)x^3$$

And

$$\alpha(a_0 + a_1x + a_2x^2 + a_3x^3) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + (\alpha a_3)x^3.$$

1.1.2 Subspaces of a Vector Space

Definition 1.1: Subspace

For any vector space, a **subspace** is a subset that is itself a vector space, under the addition and scalar multiplication.

Definition 1.2: Subspace

A **subspace** of a vector space V is a nonempty subset W of V **that has two properties:**

1. $\forall u, v \in W : u + v \in W$ (or W is closed under scalar addition).
2. $\forall u \in W, \forall \alpha \in \mathbb{R} : \alpha u \in W$ (or W is closed under scalar multiplication).

Definition 1.3: Subspace

A **subspace** of a vector space V is a nonempty subset W of V if and only if :

$$\forall u, v \in W, \forall \alpha, \beta \in \mathbb{R} : \ \alpha.u + \beta.v \in W$$

Notice:

Any vector space has a trivial subspace $\{0_V\}$. At the opposite extreme, any vector space has itself for a subspace. A subspace that is not the entire space is a **proper subspace.**

Example 1.1

$\mathbb{R}$ is a subspace of $\mathbb{R}^2$ and $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$, and so on.

Example 1.2

Let F be the subset of $\mathbb{R}^3(\subset \mathbb{R}^3)$ consisting of all vectors of the from $(a, 0, 0)$ (with zeros as second and third components and $\alpha \in \mathbb{R}$, i.e, $W = \{(a, 0, 0) \in \mathbb{R}^3\}$)

Show that F is a subspace of $\mathbb{R}^3$.

Solution:

Let $(a,0,0),(b,0,0) \in W$, and let $\alpha \in \mathbb{R}$.

We get:

1. $(a,0,0)+(b,0,0)=(a+b,0,0) \in W$
2. $\alpha(a,0,0)=(\alpha a,0,0) \in W$

W is closed under addition and scalar multiplication.

Thus, W is subspace of $\mathbb{R}^3$

Geometrically, W is set of vectors that lie on the x-axis.

Example 1.3

Let F be the set of vectors of $\mathbb{R}^3$ of the from (a,a^2,b), namely $F=\{(a,a^2,b) \in \mathbb{R}^3\}$

Show that F is not a subspace of $\mathbb{R}^3$.

Solution:

Let $(a,a^2,b),(c,c^2,d) \in F$

$(a,a^2,b)+(c,c^2,d)=(a+c,a^2+c^2,b+d) \neq (a+c,(a+c)^2,b+d)$

Since $a^2+c^2 \neq (a+c)^2$

Thus, $(a,a^2,b)+(c,c^2,d) \notin F$

Therefore, F is not subspace.

Theorem 1.1

Let F be a subspace of a vector space V. F **contains the zero vector of** V**.**

Example 1.4

Let F be the set of vectors of $\mathbb{R}^3$ of the form $(a,a,a+2)$, namely $F=\{(a,a,a+2) \in \mathbb{R}^3\}$

Show that F is not a subspace of $\mathbb{R}^3$.

Solution:

If $(a,a,a+2)=(0,0,0)$, then $a=0$ and $a+2=0$. This system is inconsistent it has no solution.

Thus $(0,0,0) \notin F$. The necessary condition does not hold.

Therefore, F is not subspace of $\mathbb{R}^3$.

Example 1.5

Vector space that are not $\mathbb{R}^n$'s also have subspaces. The space of
an n-th order polynomial $\mathcal{P}_n = \{a_0 + a_1x + a_2x^2 + ... + a_nx^n \backslash a_0, a_1, a_2...a_n \in \mathbb{R}\}$
has a subspace $\mathcal{P}_1$ comprised of polynomials

$$\{a_0 + a_1x \backslash a_0, a_1 \in \mathbb{R}\}$$

Example 1.6

Being vector space themselves, subspace must satisfy the closure conditions. The set $\mathbb{R}^+$ is not a subspace of the vector space $\mathbb{R}$ because it is not closed under scalar multiplication $x \in \mathbb{R}^+, \alpha = -1 \in \mathbb{R}$ then $-1.x \notin \mathbb{R}^+$.

Exercise:

Let $F = \{f \backslash f : \ \mathbb{R} \to \mathbb{R}\}$ the vector space of functions on $\mathbb{R}$.
Which of the following are subspaces of F ?

1. $F_1 = \{f \backslash f : \mathbb{R}, f(0) = 0\}$.
2. $F_2 = \{f \backslash f : \mathbb{R}, f(0) = 3\}$.
3. $F_3 = \{f \backslash f : \mathbb{R}, \ \text{for some} \ \ \alpha \in \mathbb{R}, f(0) = \alpha \ \text{for every} \ x\}$.

1.1.3 Linear Combination

Definition 1.4: Linear Combination

Let V be a vector space over $\mathbb{K}$. We say that the vector v is a **linear combination** of the vectors $v_1, v_2, ...v_n$ if there exist scalar $\alpha_1, \alpha_2, ...\alpha_n$ such that v can be written:

$$v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n.$$

$$\exists\ \alpha_1, \alpha_2, ...\alpha_n \in K : v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$$

Example 1.7

The vector $(7, -5)$ is a linear combination of the vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, since it can be written:

$$(7, -5) = 7(1, 0) - 5(0, 1)$$

Example 1.8

The vector $(5, 4, 2)$ is a linear combination of the vectors $(1, 2, 0), (3, 1, 4)$ and $(1, 0, 3)$, since it can be written:

$$(5, 4, 2) = (1, 2, 0) + 2(3, 1, 4) - 2(1, 0, 3)$$

1.1.4 Generating Sets

Definition 1.5: Generating set or Spanning Set

The vectors $v_1, v_2, ...v_n$ **is said to generate (or to span)a vector space** if every vector in the space can be expressed as a **linear combination of these vectors.** The set $S = \{v_1, v_2, ...v_n\}$ is called **a generating set or spanning set** and we write:

$$V = [S] = \{\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n \backslash \alpha_1, \alpha_2, ...\alpha_n \in \mathbb{K}\}$$

Examples:

- $\mathbb{R} = [\{1\}] \iff \forall x \in \mathbb{R} : x = 1.x.$
- $\mathbb{R}^2 = [\{(1, 0), (0, 1)\}] \iff \forall (x, y) \in \mathbb{R}^2, (x, y) = x.(1, 0) + y.(0, 1).$

- $\mathbb{R}^3 = [\{(1,0,0),(0,1,0),(0,0,1)\}] \iff$

$$\forall (x,y,z) \in \mathbb{R}^3, (x,y,z) = x.(1,0,0) + y.(0,1,0) + z.(0,0,1)$$

.

- The vector space of quadratic polynomials $\mathcal{P}_2$ is spanned (or generated) by the set of monomials $\{1, x, x^2\}, \mathcal{P}_2 = [\{1, x, x^2\}] \iff \forall p(x) \in \mathcal{P}_2, \exists a, b, c \in \mathbb{R} : p(x) = a + bx + cx^2$.

- $\mathbb{C} = [\{1, i\}] \iff z \in \mathbb{C} : \ z = x.1 + y.i.$

1.1.5 Linear Dependence and Independence

Definition 1.6

- The set of vectors $\{v_1, v_2, ..., v_n\}$ in a vector space V is said to be **linearly dependent** if there exist scalars $\alpha_1, \alpha_2, ..., \alpha_n$, **not all zero**, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0_V$$

$$\exists\ \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{K}, \alpha_i \neq, i \in \{1, ..., n\} \wedge \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0_V$$

- The set of vectors $\{v_1, v_2, ..., v_n\}$ is **linearly independent** if

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0_V$$

can be satisfied when

$$\alpha_1 = \alpha_2 = ... = \alpha_n = 0_{\mathbb{K}}$$

$$\forall \alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{K} : \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0_V \implies \alpha_1 = \alpha_2 = ... = \alpha_n = 0_{\mathbb{K}}$$

Example 1.9

If $\vec{u} = 2\vec{w}$ and $\vec{v} = 3\vec{w}$. Prove that the vectors, $\vec{u}$ and $\vec{v}$ are linearly dependent.

$$\vec{u} = 2\vec{w} \implies \vec{w} = \frac{1}{2}\vec{u}$$

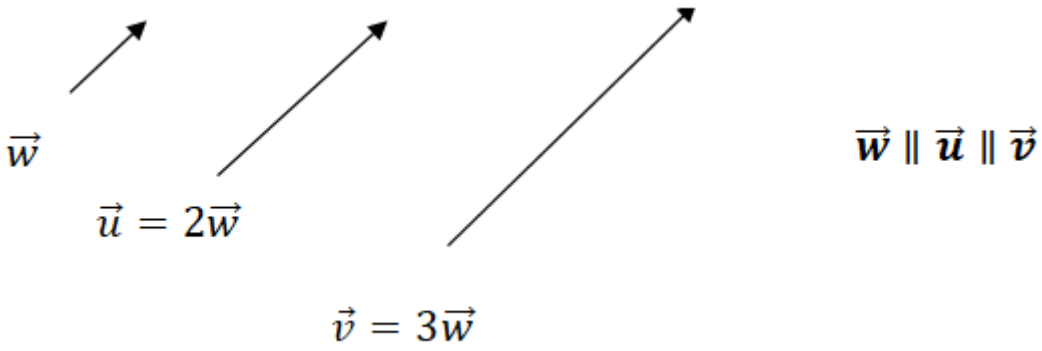
$\vec{v} = 3\vec{w} \implies \vec{v} = \frac{3}{2}\vec{u}$, that is $\vec{v}$ is scalar multiple of $\vec{u}$.

Then $\vec{u}$ and $\vec{v}$ are collinear, i.e, vectors that lie along the same line or parallel lines.

They are also known as parallel vectors. $\vec{v} = \frac{3}{2}\vec{u} \implies \vec{v} - \frac{3}{2}\vec{u} = \vec{0}$. One obtains, $2\vec{v} - 3\vec{u} = \vec{0}$.

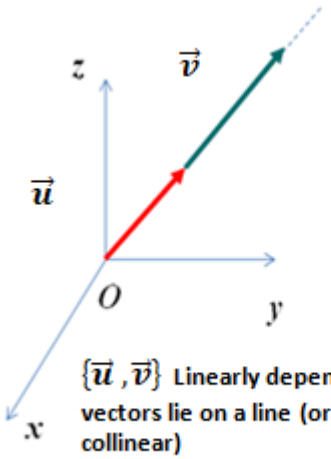
We conclude that $\vec{u}$ and $\vec{v}$ are linearly dependent.

Solution:

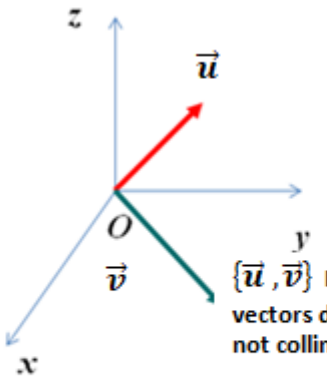


Example 1.10

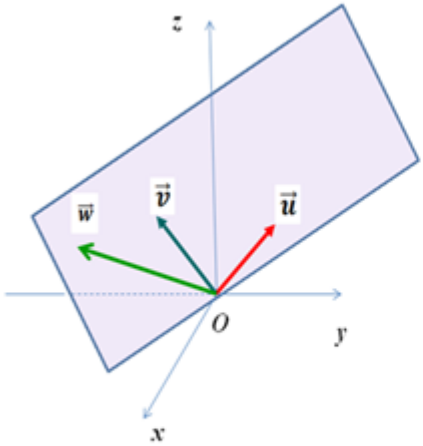
Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$



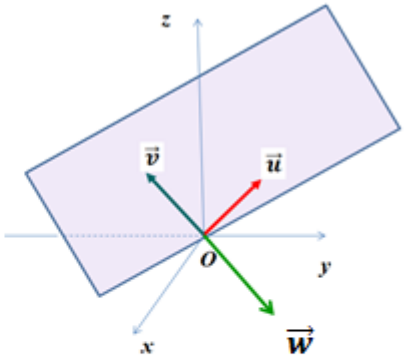
$\{\vec{u}, \vec{v}\}$ Linearly dependent, vectors lie on a line (or collinear)



$\{\vec{u}, \vec{v}\}$ Linearly independent, vectors do not lie on a line (or dot not collinear)



$\{\vec{u}, \vec{v}, \vec{w}\}$ Linearly dependent, vectors lie in a plane



$\{\vec{u}, \vec{v}, \vec{w}\}$ Linearly independent, vectors do not lie in a plane

Example 1.11

Show that the set $\{(3,-2,2),(3,-1,4),(1,0,5)\}$ is linearly independent in $\mathbb{R}^3$.

Solution:

Suppose that $\alpha_1(3,-2,2)+\alpha_2(3,-1,4)+\alpha_3(1,0,5)=(0,0,0)$

$$\Longrightarrow (3\alpha_1,-2\alpha_1,2\alpha_1),(3\alpha_2,-1\alpha_2,4\alpha_2),(1\alpha_3,0\alpha_3,5\alpha_3)(0,0,0)$$

$$\implies (3\alpha_1 + 3\alpha_2 + \alpha_3, -2\alpha_1 - \alpha_2, 2\alpha_1 + 4\alpha_2 + 5\alpha_3)$$

$$\implies \begin{cases} 3\alpha_1 + 3\alpha_2 + \alpha_3 = 0 \\ -2\alpha_1 - \alpha_2 = 0 \\ 2\alpha_1 + 4\alpha_2 + 5\alpha_3 = 0 \end{cases} \implies \begin{cases} \alpha_2 = -2\alpha_1 \\ \alpha_3 = 3\alpha_1 \\ \alpha_1 = \alpha_3 \end{cases}$$

This system has the **unique** solution $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.

Example 1.12

Consider the function $f(x) = x^2 + 1, g(x) = 3x - 1, h(x) = -4x + 1$ of the vector space $\mathcal{P}_2$ of polynomials of degree ≤ 2.

Show that the set of the function $\{f, g, h\}$ is linearly independent.

Solution:

Suppose $\alpha_1 f + \alpha_2 g + \alpha_3 h = 0$.

Since for any real number, $\alpha_1 f(x) + \alpha_2 g(x) + \alpha_3 h(x) = 0 \implies \alpha_1(x^2 + 1) + \alpha_2(3x - 1) + \alpha_3(-4x + 1) = 0$.

Consider three convenient values of x.

We get:

$x = 0 \implies \alpha_1 + \alpha_3 = 0$

$x = 1 \implies 2\alpha_2 - 3\alpha_3 = 0$

$x = -1 \implies 2\alpha_1 - 4\alpha_2 - 5\alpha_3 = 0$

$$\implies \begin{cases} \alpha_1 - \alpha_2 + \alpha_3 = 0 \\ 2\alpha_2 - 3\alpha_3 = 0 \\ 2\alpha_1 - 4\alpha_2 - 5\alpha_3 = 0 \end{cases} \implies \begin{cases} \alpha_3 = -\alpha_1 + \alpha_2 \\ 5\alpha_1 - \alpha_2 = 0 \\ 7\alpha_1 - 9\alpha_2 = 0 \end{cases}$$

It can be show that this system of three equations has the unique solution $\alpha_1 = 0, \alpha_2 = 0$ and $\alpha_3 = 0$.

Thus, $\alpha_1 f + \alpha_2 g + \alpha_3 h = 0 \implies \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$.

The set $\{f, g, h\}$ is linearly independent.

Properties:

- A set consisting of two or more vector space is linearly dependent if and only if is possible to express one of the vectors as linearly combination of other vectors.
- Let V be a vector space. any set of vectors in V that contains the zero vector is linearly dependent.
- Let the set $\{v_1, v_2, ..., v_n\}$ be linearly dependent in a vector space V. Any set of vectors in V that contains these vectors will also be linearly dependent.

1.1.6 Bases and Dimension:

Definition 1.7: Basis

A finite set of vectors $S = \{v_1, v_2, ...v_n\}$ is called a **basis** for a vector space V if:

1. The set S **spans (generates) the vectors space.**
2. The set S is **linearly independent.**

Definition 1.8: Standard Basis

The set of n-vectors $\{e_1, e_2, ..., e_n\}$ is a basis for $\mathbb{R}^n$

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), e_n = (0, 0, ..., 1)$$

This basis is called **standard basis** for $\mathbb{R}^2$.

Example 1.13

Show that the set $\{(1, 2), (3, 4)\}$ is a basis for $\mathbb{R}^2$.

Solution:

$\{(1, 2), (3, 4)\}$ Spans the space ?

Let (x_1, x_2) be an arbitrary of $\mathbb{R}^2$.

Suppose $(x_1, x_2) = \alpha_1.(1, 2) + \alpha_2.(3, 4)$

$$\Longrightarrow \begin{cases} x_1 = \alpha_1 + 3\alpha_2 \\ x_2 = 2\alpha_1 + 4\alpha_2 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = x_1 - \dfrac{x_2}{2} \\ \alpha_2 = -2x_1 + \dfrac{3x_2}{2} \end{cases}$$

Thus, the set spans the space $\mathbb{R}^2$.

$\{(1,2),(3,4)\}$ is linearly independent ?

Consider the identity $\beta_1.(1,2) + \beta_2.(3,4) = (0,0)$

$$\begin{cases} \beta_1 + 3\beta_2 = 0 \\ 2\beta_1 + 4\beta_2 = 0 \end{cases} \Longrightarrow \beta_1 = 0 \text{ and } \beta_2 = 0 \text{ is the unique solution.}$$

Thus, the set is linearly independent.

$\{(1,2),(3,4)\}$ Spans $\mathbb{R}^2$ and is linearly independent. It forms a basis for $\mathbb{R}^2$.

Example 1.14

Show that $\{f,g,h\}$. Where $f(x) = x^2 + 1, g(x) = 3x - 1$ and $h(x) = -4x + 1$ is a basis of $\mathcal{P}_2$.

Solution:

We have already proven in the previous section (Example 1.4), that the set of the functions$\{f,g,h\}$ is linearly independent.

Is it spanning (or generator) set of $\mathcal{P}_2$?

p is thus a polynomial of the form $p(x) = a + bx + cx^2$.

Suppose $p(x) = \alpha f(x) + \beta g(x) + \gamma h(x)$ for some scalar α, β, γ.

This gives

$$a + bx + cx^2 = \alpha f(x) + \beta g(x) + \gamma h(x)$$

$$= \alpha(x^2 + 1) + \beta(3x - 1)\gamma(-4x + 1)$$

$$= (\alpha - \beta + \gamma) + (3\beta - 4\gamma) + \alpha x^2$$

$$\Longrightarrow \begin{cases} a = \alpha - \beta + \gamma \\ b = 3\beta - 4\gamma \\ c = \alpha \end{cases} \Longrightarrow \begin{cases} \alpha = c \\ \beta = 4a - 4c - b \\ \gamma = 3a - 3c - b \end{cases}$$

Thus, the polynomial p can be expressed $p(x) = \alpha f(x) + \beta g(x) + \gamma h(x)$.

The functionsf, g, and h span $\mathcal{P}_2$.

They form a basis for $\mathcal{P}_2$.

Properties:

- Let $\{v_1, v_2, ..., v_n\}$ be a basis for a vector space V.

  If $\{u_1, u_2, ..., u_m\}$ is a set of more than n $(m > n)$ vectors in V, then this set linearly dependent.

- Any two bases for a vector space V consist of same number of vectors.

Definition 1.9 (Dimension)

If a vector space V has a **basis consisting of** n vectors, then the **dimension of** V is said to be n. We write $dim(V)$ for the dimension of V.

Example 1.15

$dim(\mathbb{R}^2) = 2$

$dim(\mathbb{R}^3) = 3$

$dim(\mathbb{R}^n) = n$

$dim(\mathbb{C}) = 2$

$dim\{(0, 0, ..., 0)\} = 0$

$dim(\mathcal{P}_n) = n + 1$

Example 1.16

The one-dimensional subspaces of $\mathbb{R}^3$ are lines through the origin.

Example 1.17

The two-dimensional subspaces of $\mathbb{R}^3$ are plans through the origin.

Example 1.18

Find a basis and the dimension of subspace W of $\mathbb{R}^3$. Where $W = \{(x, y, z) \backslash x + 2z = 0\}$?

Solution:

Spanning set

$W = \{(x, y, z) \backslash x + 2z = 0\}$

$W = \{(x, y, z) \backslash x = -2z\}$

$W = \{(-2z, y, z) \backslash y, z \in \mathbb{R}\}$

$W = \{-z(-2, 0, 1) + y(0, 1, 0) \backslash y, z \in \mathbb{R}\}$

$W = [\{(-2, 0, 1), (0, 1, 0)\}]$

Linearly dependent

$$\alpha(-2, 0, 1) + \beta(0, 1, 0) = (0, 0, 0) \implies \begin{cases} -2\alpha = 0 \\ \beta = 0 \\ \alpha = 0 \end{cases}$$

The system has a unique solution $\alpha = 0$ and $\beta = 0$.

Hence the subset $\{(-2, 0, 1), (0, 1, 0)\}$ is a basis of W and $dim(W) = 2$.

Properties:

Let V be a vector space of dimension n.

- If $S = \{v_1, v_2, ..., v_n\}$ is a set of n linearly dependent vectors in V, then S is basis for V.
- If $S = \{v_1, v_2, ..., v_n\}$ is a set of n vectors that spans (or generate) V, then S is a basis for V.

Example 1.19

Prove that the set $S = \{(1, 3, -1), (2, 1, 0), (4, 2, 1)\}$ is a basis for $\mathbb{R}^3$.

Solution:

Since $dim(\mathbb{R}^3) = |S| = 3$. It suffices to show that this set linearly independent or it spans (or generates) $\mathbb{R}^3$.

Let us check for linear independence.

Suppose

$$\alpha_1(1,3,-1) + \alpha_2(2,1,0) + \alpha_3(4,2,1) = (0,0,0)$$

$$\implies \begin{cases} \alpha_1 + 2\alpha_2 + 4\alpha_3 = 0 \\ 3\alpha_1 + \alpha_2 + 2\alpha_3 = 0 \\ -\alpha_1 + \alpha_3 = 0 \end{cases}$$

This system has the unique solution $\alpha_1 = 0, \alpha_2 = 0$ and $\alpha_3 = 0$.

Thus the vectors are linearly independent.

The set $S = \{(1,3,-1),(2,1,0),(4,2,1)\}$ is therefore a basis for $\mathbb{R}^3$.

Properties:

Let W is subspace of finite dimensional vector space. Then:

- W is also finite dimensional and $dim(W) \leq dim(V)$.
- $dim(W) = dim(V)$ if and only if $W = V$.

Results:

Let V_1 and V_2 be two subspace of vector V. Then:

- Their intersection i.e, $V_1 \cap V_2$ is a subspace of V.
- Their union $V_1 \cup V_2$ need not be a subspace of V.
- Their sum, defined as $V_1 + V_2 = \{x_1 + x_2 \backslash x_1 \in V_1, x_2 \in V_2\}$ is a subspace of V.
- $dim(V_1 + V_2) = dim(V_1) + dim(V_2) - dim(V_1 \cap V_2)$.
- The sum $V_1 \oplus V_2$ is called direct sum of V, $V = V_1 \oplus V_2$, if $V_1 \cap V_2 = \{0\}$. Every vector of V can be written in a unique way as $x_1 + x_2$ where $x_1 \in V_1, x_2 \in V_2$.
- $dim(V_1 \oplus V_2) = dim(V_1) + dim(V_2)$.

Example 1.20

Let $V = \mathbb{R}^2$, $V_1 = \{(x, x) : x \in \mathbb{R}\}$ and $V_2 = \{(x, -x) : x \in \mathbb{R}\}$.

Then $\mathbb{R}^2 = V_1 + V_2$. Indeed $(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right) + \left(\frac{x-y}{2}, -\frac{x-y}{2}\right)$

Example 1.21

Let $V = \mathbb{R}^2, V_1 = \{(x, 2x) : x \in \mathbb{R}\}, V_2 = \{(x, 3x) : x \in \mathbb{R}\}$.

Then $\mathbb{R}^2 = V_1 \oplus V_2$.

Indeed $(x, y) = (a, 2a) + (b, 3b)$

$\implies (x, y) = (3x - y, 6x - 2y) + (y - 2x, 3y - 6x)$.

Let $(x, y) \in V_1 \cap V_2 \implies (x, y) = (a, 2a) = (b, 3b)$ for some $a, b \in \mathbb{R}$. By comparing the first component we get $a = b$ and comparing the second we get $a = 0$.

1.2 Linear Maps

1.2.1 Definitions and Basic Properties

Definition 1.10: Linear map or linear transformation

Suppose V and W are vectors spaces over the same field $\mathbb{K}$.

A map $f : V \to W$ is called **linear map** or **linear transformation** or **homomorphism** if f **preserves structure**.

If $u, v \in V, \alpha \in \mathbb{K}$ then

$$f(u+v) = f(u) + f(v) \textbf{ and } f(\alpha u) = \alpha f(u)$$

Property:

For any map $f : V \to W$ between vector space these statement are equivalent:

1. f **preserves structure**

$$f(u+v) = f(u) + f(v) \text{ and } f(\alpha u) = \alpha f(u)$$

2. f **preserves linear combinations of two vectors**

$$\forall u, v \in V : \forall \alpha, \beta \in \mathbb{K} : f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$$

Example 1.22

Let : $\mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (x + y, x - y)$.

Show that f is linear map:

$\forall X, Y \in \mathbb{R}^2, \forall \alpha, \beta \in \mathbb{R} : f[\alpha X + \beta Y] = f[\alpha(x_1, y_1) + \beta(x_2, y_2)]$

$$= f[(\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)]$$

$$= f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2, \alpha x_1 + \beta x_2 - \alpha y_1 + \beta y_2)$$

$$= (\alpha x_1 + \alpha y_1, \alpha x_1 - \alpha y_1) + (\beta x_2 + \beta y_2, \beta x_2 - \beta y_2)$$

$$\alpha f(x_1, y_1) + \beta f(x_2, y_2)$$

Thus $f[\alpha X + \beta Y] = \alpha f(X) + \beta f(Y)$

Therefore, f is a linear map.

Example 1.23

Show that $f : (x, y) \mapsto (x + y + 1, x - y)$ is not a linear map.

$f[\alpha X + \beta Y] = f[\alpha(x_1, y_1) + \beta(x_2, y_2)]$

$f(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + 1, \alpha x_1 + \beta x_2 - \alpha y_1 + \beta y_2)$

$\alpha f(X) + \beta f(Y) = \alpha(x_1 + y_1 + 1, x_1 + y_1) + \beta(x_2 + y_2 + 1, x_2 - y_2)$

$= (\alpha x_1 + \alpha y_1 + \alpha, \alpha x_1 - \beta y_1) + (\beta x_1 + \beta y_1 + \beta, \beta x_1 - \beta y_1)$

$= (\alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + \alpha + \beta, \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2)$

$$f[\alpha X + \beta Y] \neq \alpha f(X) + \beta f(Y)$$

Example 1.24

For any vector space V, W over $\mathbb{R}$, $f : V \to W$.

Defined by $x \mapsto 0$ for all $x \in V$ is linear map and it is called the **zero map (or zero transformation).**

Example 1.25

$Id_V : V \to V$ defined by: $x \mapsto x$ for all $x \in V$ is a linear map and it is called the **identity operator.**

Example 1.26

For each $1 \leq i \leq n, p_i : \mathbb{R}^n \to \mathbb{R}$ defined by:

$p_i(x_1, x_2, ..., x_n) = x_i$ is a linear map it is called the i**-th projection.**

Example 1.27

The function $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \sin x$ is not linear map.

Since

$$\sin(x_1 + x_2) \neq \sin x_1 + \sin x_2$$

$$\sin\left(\frac{\pi}{2} + \frac{\pi}{3}\right) \neq \sin\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{3}\right) \implies f(x_1 + x_2) \neq f(x_1) + f(x_2).$$

Properties:

Here are some simple properties of linear map $f : V \to W$, though which the map maintains the structure of vector space

1. $f(0_V) = 0_W$ The image of the null vector is a null vector.
2. $f(-x) = -f(x)$ The image of opposite vector is an opposite image of the vector.
3. If V_1 is a subspace of V, then $f(V_1)$ is a subspace of W.
   The image of subspace of V is a subspace of W.
   If W_1 is a subspace of W, then $f^{-1}(W_1)$ is a subspace of V.
4. The composite map of two linear maps is a linear map.

1.2.2 The Image Space and Null Space

Definition 1.11: The image space or range space

The **image space (or range space)** of f denoted by Imf or $f(V)$ is the set of all vectors $f(v)$ in W for some $v \in V$:

$$Imf = f(V) = \{f(v) : v \in V\} \subset W$$

Definition 1.12: The null space or kernel of a linear map

The **null space** (or **kernel**) of f denoted by $kerf$, is the set of all vectors v in V such that $f(v) = 0_W$ (the zero vector in W):

$$kerf = \{v \in V : f(v) = 0_W\} = f^{-1}(\{0_W\}) \subset V$$

Example 1.28

Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by:

$f(x, y, z) = (x + y - z, x - y + z, y - z)$

Show $kerf$ and Imf .

Solution:

$$kerf = \{v \in \mathbb{R}^3 : f(v) = 0_{\mathbb{R}^3}\}$$

Span

$$kerf = \{(x, y, z) \in \mathbb{R} : (x + y - z, x - y + z, y - z) = (0, 0, 0)\}$$

$$= \{(x, y, z) \in \mathbb{R} : x + y - z = 0, x - y + z = 0, y - z = 0\}$$

$$\implies kerf = \{(x, y, z) \in \mathbb{R}^3 : x = 0, y = z\}$$

$$\implies kerf = \{(0, y, y), y \in \mathbb{R}\} = \{y(0, 1, 1) : y \in \mathbb{R}\}$$

Hence $kerf = [\{(0, 1, 1)\}]$.

Linearly independent

Since $(0, 1, 1) \neq (0, 0, 0)$, the set of the unique vector $\{(0, 1, 1)\}$ is the basis of $kerf$

$$Imf = f(V) = \{f(v) : v \in \mathbb{R}^3\}$$

Span

$$Imf = \{(x + y - z, x - y + z, y - z) : (x, y, z) \in \mathbb{R}^3\}$$

$$= \{x(1, 1, 0) + (y - z)(1, -1, 1) : (x, y, z) \in \mathbb{R}^3\}$$

$$Imf = [\{(1, 1, 0), (1, -1, 1)\}]$$

Linearly independent

$\alpha(1,1,0)+\beta(1,-1,1)=(0,0,0)$

$$\begin{cases}\alpha+\beta=0\\ \alpha-\beta=0\\ \beta=0\end{cases}\implies\begin{cases}\alpha=0\\ \beta=0\end{cases}$$ Hence, the set $\{(1,1,0),(1,-1,1)\}$ is linearly independent.

Therefor, $\{(1,1,0),(1,-1,1)\}$ is the basis of the image.

1.2.3 Linear Maps and Dimensions

Properties

Let $f:V\to W$ be a linear map

1. The kernel of f is a subspace of V.
2. The image of f is a subspace of W.
3. $dim(V)=dimkerf+dimImf$.
4. f is injective if and only if $kerf=\{0_V\}$.
5. f is surjective if and only if $Imf=f(V)=W$.
6. If $dim(V)=dim(W)$ then f injective $\iff$ f surjective $\iff$ f bijective.
7. The *rank* of f is the dimension of $f(V)$: $rankf=dimImf$.

Example 1.29

Let

$$\begin{array}{rcl} f:\mathbb{R}^2 & \to & \mathbb{R}\\ (x,y) & \mapsto & f(x,y)=x-y\end{array}$$

Show the $rankf$ and $dimkerf$. What do you conclude ?

$$rankf=dimImf$$

We first check the $Imf = f(\mathbb{R}^2) = \{f(v) : v \in \mathbb{R}\}$

$$Imf = \{xy, (x, y) \in \mathbb{R}^2\} = \mathbb{R}$$

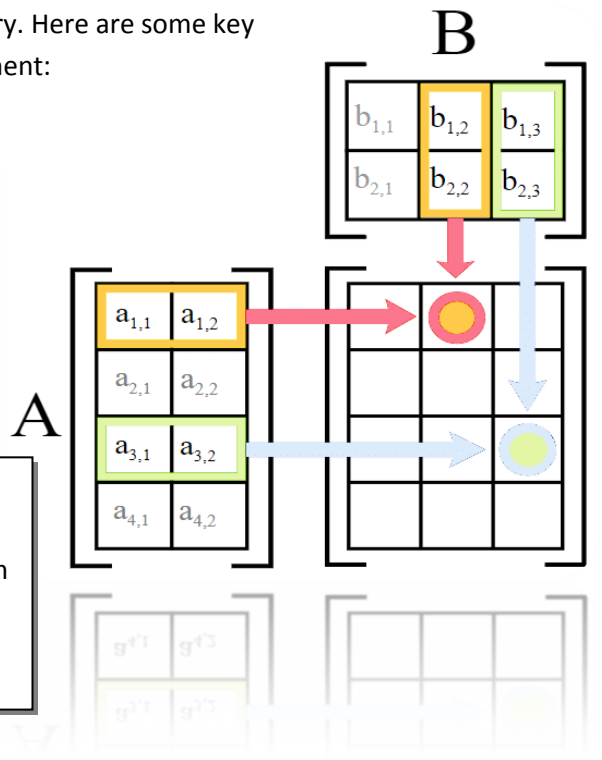
$rankf = dimImf = dim(\mathbb{R}) = 1 \implies f$ is surjective.

$dim(\mathbb{R}^2) = dimkerf + dimImf \implies dim(kerf) = 2 - rankf = 1$

$dimkerf \neq 0 \implies f$ is not injective.

Matrices and Determinants

Matrices and determinants are fundamental to linear algebra and have been shaped by influential mathematicians throughout history. Here are some key figures associated with their development:



Arthur Cayley (1821–1895): Cayley made significant contributions to matrix theory, including introducing the term "matrix" and studying matrix algebra. He also developed the concept of determinants.

James Joseph Sylvester (1814–1897): Sylvester worked extensively on matrices and determinants, contributing to their theory and applications. He introduced the term "matrix" independently of Cayley and made important contributions to the theory of invariants.

Évariste Galois (1811–1832): Galois made foundational contributions to the theory of determinants, particularly in his study of permutations and their applications to solving polynomial equations.

Carl Friedrich Gauss (1777–1855): Gauss made significant contributions to the theory of determinants, including the development of methods for solving systems of linear equations using determinants.

David Hilbert (1862–1943): Hilbert's work on invariant theory and the theory of algebraic forms contributed to the understanding of determinants and their properties.

Arthur Schönflies (1853–1928): Schönflies made important contributions to the study of determinants, particularly in his work on the theory of linear transformations and their representation by matrices.

William Rowan Hamilton (1805–1865): Hamilton's work on quaternions and their relationship to matrices and determinants contributed to the development of linear algebra.

Chapter 2

Matrices and Determinants

2.1 Matrices

In this chapter's lectures, we deal about matrices. **Matrix(plural, matrices)** is an array of numbers that are **arranged in rows and columns** and enclosed in parentheses. Both A and B are examples of matrix.

$$A = \begin{bmatrix} 2 & -3 & \sqrt{5} \\ 7 & 8 & 0 \end{bmatrix}, \; B = \begin{bmatrix} -2 & 3 & 2 \\ 0 & 1 & 4 \\ 4 & 7 & 9 \end{bmatrix}$$

Why matrices?

Consider the following set of equations:

$\begin{cases} x + y = 7 \\ 3x - y = 5 \end{cases}$, it is easy to show that $x = 3$ and $y = 4$.

How about solving

$$\begin{cases} x + y - 2z = 7 \\ 2x - y - 4z = 2 \\ -5x + 4y + 10z = 1 \\ 3x - y - 6z = 5 \end{cases}$$



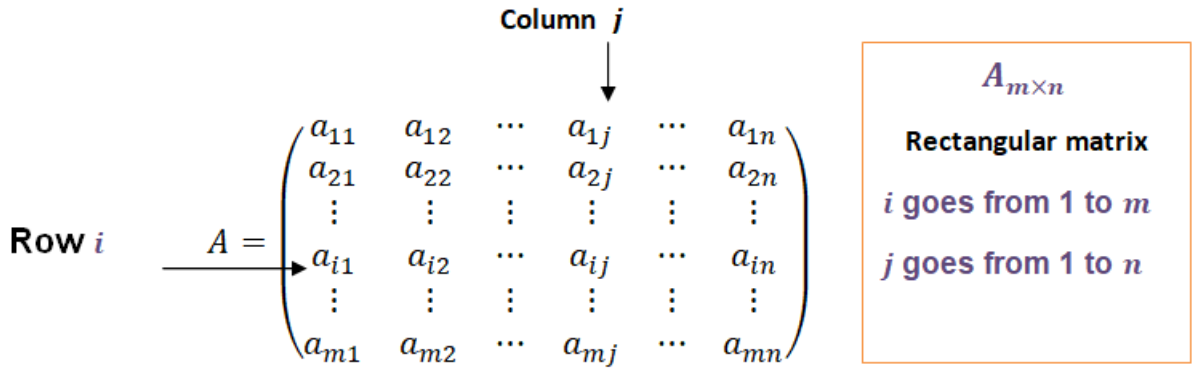
Matrices can help.....

Matrices arise in the study of **simultaneous linear equations** and the study of **linear Algebra**. Any **linear map** (or **linear transformation**) even can be represented via a matrix. One can combine matrices according to a set of matrix operations. Indeed, matrices have become a critical tool in representing, manipulating and used to fast solving of incredibly large **linear systems** that occur in such fields as **dynamical systems partial differential equations, graph theory computer graphics** and many more. We will define matrices and how to add and multiply them, discuss some special matrices such as the **identity** and **zero matrix**, learn about **transpose** and **inverse matrix**.

2.1.1 Types of Matrices

Rectangular Matrix

An $m \times n$ matrix or an m-by-n matrix is rectangular array of numbers (real or complex) with m **rows** and n **columns** it is called matrix of **order (or size)** $m \times n$:



We also use the more brief notation:

$$A = (a_{ij}), (i = 1, ..., m; j = 1, ..., n)$$

Here, the matrix element of A in the i^{th} row and the j^{th} column is denoted as a_{ij}. The set of all $m \times n$ matrices is denoted by $\mathcal{M}_{m\times n}(K)$ or $\mathcal{M}_{m,n}(K)$.

Example 2.1

The objects below are matrices:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{2\times 3}, \; B = \begin{pmatrix} 2+2i & \frac{1}{4} \\ e & \pi \end{pmatrix}_{2\times 2}, \; C = \begin{pmatrix} e^{-2t} & \cos(3t) \\ 2 & -3 \\ \sin(3t) & e^{-t} \end{pmatrix}_{3\times 2},$$

$$D = \begin{pmatrix} 3 & 1 & -4 \\ 2 & 4 & 2 \\ -1 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}_{4\times 3}$$

Null Matrix (Zero Matrix)

The **null matrix** (or the zero matrix), denoted by $0_{m\times n}$, is a matrix consisting of all zero elements $a_{ij} = 0, \; \forall i$ and j.

$$0_{m\times n} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Null Matrix (Zero Matrix)

Row Matrix and Column Matrix

A matrix with just one row (a **row matrix**) or just one column (a **column matrix**) can be regarded as a vector. A vector of order n can be viewed either as a $1 \times n$ matrix or as an $n \times 1$ matrix. The entries of the vector is called **components**.

Example 2.2

When $n = 3$. We would write a column vector as

$$A = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

and a row vector as

$$B = \begin{pmatrix} a & b & c \end{pmatrix}$$

Equality of Matrices

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal if they have the same number of rows, the same number of columns and corresponding elements are equal. Equality of matrices does not hold of the mentioned properties does not hold.

$$A = B \iff \begin{cases} (i = 1, ..., m;\ j = 1, ..., n) \\ a_{ij} = b_{ij} \end{cases}$$

Example 2.3

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\ ,\ B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

If $A = B$ then $a = 1, b = 2, c = 3, d = 4$.

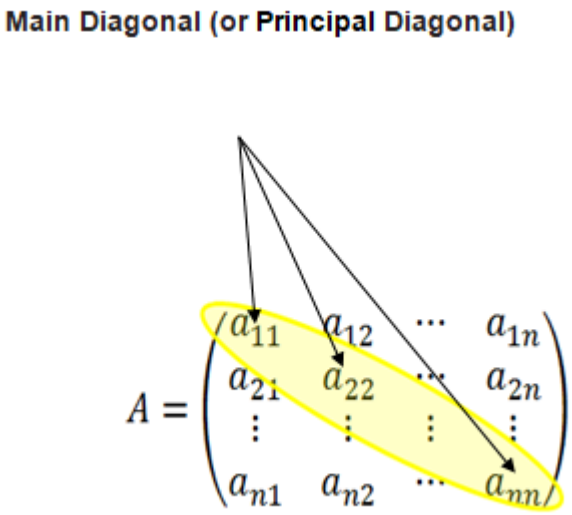
Square Matrices

A matrix is **square** if the number of rows equals the number of columns $m = n$. The set of all square matrices is called $n \times n$ matrices (or matrices of order n) is denoted by $\mathcal{M}_{n\times n}(\mathbb{R})$ or $\mathcal{M}_n(\mathbb{R})$.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix}$$

Square Matrix

The **main diagonal(or principal diagonal)** elements of a square matrix are those elements where the row and column index are the same $a_{11}, a_{22}, ..., a_{nn}$. The other elements are non-diagonal elements. The diagonal elements form the diagonal of the matrix:



2.1.2 Square Matrices

Upper Triangular Matrix

The **upper triangular matrix** is a square matrix having all the elements zero below the principal diagonal.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_{nn} \end{pmatrix}$$

Upper Triangular Matrix

Lower Triangular Matrix

The **lower triangular matrix** is a square matrix having all the elements zero above the principal diagonal.

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

Lower Triangular Matrix

Diagonal Matrix

The **diagonal matrix** is a square matrix that **non-zero element in principal diagonal** and zero in all other positions.

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}$$

Diagonal Matrix

Scalar Matrix

The **scalar matrix** is a diagonal matrix in with all the elements on principal diagonal are same.

$$A = \begin{pmatrix} \alpha & 0 & \dots & 0 \\ 0 & \alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha \end{pmatrix}$$

Scalar Matrix

Unit or Identity Matrix

Identity matrix is a diagonal matrix having each diagonal element equal to one **1**. Identity matrix of the order n is denoted bu I_n:

$$I_n = \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ \vdots & \vdots & 1 & \vdots \\ 0 & 0 & ... & 1 \end{pmatrix}$$

Identity Matrix

Example 2.4

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The matrix A is diagonal matrix.

$$I_1 = (1),\ I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$I_1,\ I_2$ and I_3 are identity matrices of order $1, 2$ and 3 respectively.

2.1.3 Matrix Operations

There are three basic arithmetic operations one can perform on a matrix or on pair of matrices of the same order (that is, two matrices with the same number of rows and the same number of columns).

Matrix Addition and Subtraction

We can only **add** and **subtract** matrices $A = (a_{ij})$ and $B = (b_{ij})$ that **have exact the same dimensions** $m \times n$.We perform the **addition** or **subtraction** of corresponding elements.

$$A \pm B = (a_{ij} \pm b_{ij})$$

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & b_{12} & ... & b_{1n} \\ b_{21} & b_{22} & ... & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & ... & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & ... & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & ... & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & ... & a_{mn} \pm b_{mn} \end{pmatrix}$$

Example 2.5

Use the matrices blow and perform the addition and subtraction of matrices (where possible).

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \; C = \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 4 & 3 \\ 1 & -1 \\ -2 & 0 \end{pmatrix}$$

Solution:

The defined addition and subtraction operations are given by:

$$A_{(3\times3)} \pm A_{(3\times3)}, \; C_{(3\times3)} \pm C_{(3\times3)}, \; A_{(3\times3)} \pm C_{(3\times3)}, \; D_{(3\times3)} \pm D_{(3\times3)}, \; B_{(3\times3)} \pm B_{(3\times3)}.$$

Let's calculate $A + C$, $A - C$ and $B + B$. However, the students are required to do the other defined operations.

$$A_{(3\times3)} + C_{(3\times3)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1+4 & 1-1 & 2+1 \\ 0+1 & -1+2 & 1-4 \\ 1-2 & 2+1 & 1+3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 0 & 3 \\ 1 & 1 & -3 \\ -1 & 3 & 4 \end{pmatrix}$$

$$A_{(3\times3)} - C_{(3\times3)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} - \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1-4 & 1-(-1) & 2-1 \\ 0-1 & -1-2 & 1+4 \\ 1+2 & 2-1 & 1-3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 2 & 1 \\ -1 & -3 & 5 \\ 3 & 1 & -2 \end{pmatrix}$$

$$B_{(3\times3)} + B_{(3\times3)} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 2+2 & -2-2 & 1+1 \\ 1+1 & -1-1 & 3+3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & 2 \\ 2 & -2 & 6 \end{pmatrix}$$

Scalar Multiplication

To multiply a matrix by a real number λ, multiply each entry of the matrix by that number. In general, the formula for the scalar product λA is:

$$\lambda A = (\lambda a_{ij}) = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}$$

Example 2.6

If $A = \begin{pmatrix} 2 & 1 \\ -5 & 2 \\ -9 & 8 \end{pmatrix}$

We want to find what $2A$ is. Then what we do is:

$$2A = \begin{pmatrix} 2.2 & 2.1 \\ 2.(-5) & 2.2 \\ 2.(-9) & 2.8 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ -10 & 4 \\ -18 & 16 \end{pmatrix}$$

Example 2.7

Perform the operations on the following matrices:

$$A = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix},\ B = \begin{pmatrix} -5 & 7 \\ 8 & 12 \end{pmatrix},\ C = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & \frac{1}{2} \\ -\frac{3}{2} & 1 & -5 \end{pmatrix},\ D = \begin{pmatrix} -1 & -2 & 5 \\ 5 & 6 & -1 \\ -1 & \frac{2}{5} & 0 \end{pmatrix}$$

a) $A + B$.

b) $2C + (-5)D$.

Solution:

a) We add the two matrices by adding their corresponding entries, so that

$$A + B = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix} + \begin{pmatrix} -5 & 7 \\ 8 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 5 & 12 \end{pmatrix}$$

b) To evaluate this expression, we first multiply each entry of the matrix C by 2 and each entry of the matrix D by -5. Then we add the resulting matrices. This gives.

$$2C + (-5)D = 2\begin{pmatrix} 1 & 2 & 4 \\ 3 & 0 & \frac{1}{2} \\ -\frac{3}{2} & 1 & -5 \end{pmatrix} + (-5)\begin{pmatrix} -1 & -2 & 5 \\ 5 & 6 & -1 \\ -1 & \frac{2}{5} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 & 8 \\ 6 & 0 & 1 \\ -3 & 2 & -10 \end{pmatrix} + \begin{pmatrix} 5 & 10 & -25 \\ -25 & -30 & 5 \\ 5 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 7 & 14 & -13 \\ -19 & -30 & 6 \\ 2 & 0 & -10 \end{pmatrix}$$

Vectors Multiplication

A **Vector multiplication** is fundamental skill to solve **Matrices multiplication**. Since rectangular matrices are composed of vectors, we will first discuss vector products, before defying the product of these "larger" matrices. The most important product of two vectors is their **dot product**. The **dot product** of two vectors provides a natural way to obtain a single numerical value from two separate lists of numbers.

Definition 2.1: The dot product or scalar product or inner product

The **dot product** (or **scalar product** or **inner product**) of two n-component real vectors is the linear combination of their components.

$$\vec{u}.\vec{v} =< \vec{u}, \vec{v} >= u_1v_1 + u_2v_2 + ... + u_nv_n$$

Most important is **the dot product** is always seen as the product of a **row vector** times a **column vector**, and its result is a 1×1 matrix (i.e, a **scalar**).

$$(x_1, x_2, ..., x_n).\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + ... + x_ny_n$$

Matrix Multiplication

Two matrices A and B must be **conformable** for multiplication to be possible i.e, **the number of columns of A must equal the number of rows of B.**

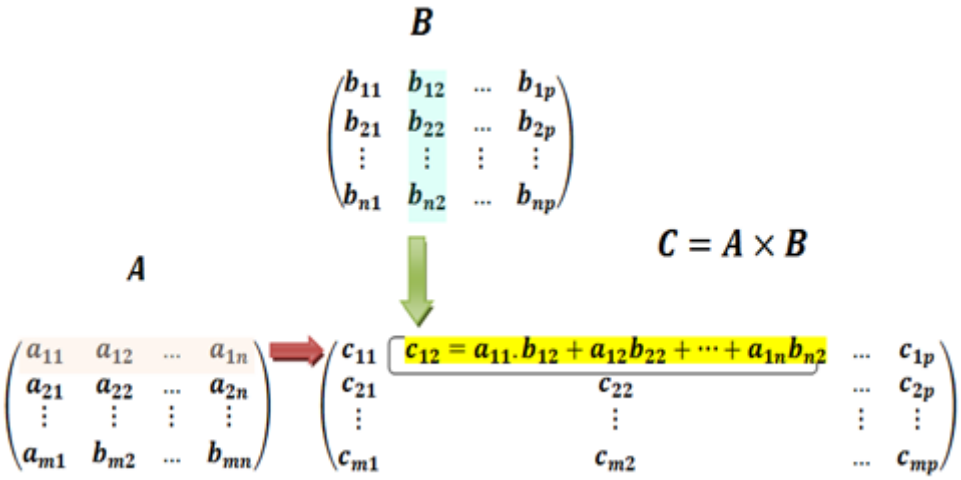
If A is an $m \times n$ matrix and B an $n \times p$ matrix, then the matrix product $A \times B$ is the $m \times p$ matrix, denoted by C ($C = A \times B$), **whose the element c_{ij} is the dot product of the i^{th} row of A with the j^{th} column of B** . We have:

$$A_{(m\times n)} \times B_{(n\times p)} = C_{(m\times p)}$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$$

Successive multiplication of row i of A with column j of B Row by Column Multiplication.

In the following diagram, we illustrate the product of two matrices A and B, showing how each intersection in the product matrix corresponds to a row of A and a column of B



Example 2.8

$A_{(2\times 1)} \times B_{(2\times 1)}$ Not possible.

$A_{(3\times 1)} \times B_{(3\times 5)}$ Not possible.

$A_{(3\times 2)} \times B_{(2\times 3)} = C_{(3\times 3)}$.

$A_{(1\times 4)} \times B_{(4\times 1)} = C_{(1\times 1)}$.

Example 2.9

If $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 3 & 5 \\ 2 & -1 \end{pmatrix}$, then $A.B$ is the 2×2 matrix

$$A.B = C = \begin{pmatrix} 2.1 + 0.3 + 1.2 & 2.3 + 0.5 + 1.(-1) \\ 1.1 + 3.3 + 5.2 & 1.3 + 3.5 + 5.(-1) \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 20 & 13 \end{pmatrix}$$

In this example, the product $B.A$ is not defined.

Remember also: $C.I = C$

$$\begin{pmatrix} 4 & 5 \\ 20 & 13 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 20 & 13 \end{pmatrix}$$

Remarks:

- The identity matrix is special because it is the **multiplication identity element** for matrices. The basic idea is that if you multiply a matrix by the identity matrix, you get the original matrix. The identity matrices play the role of the number **one**, then:

$$A.I_n = I_n.A = A$$

- The null square matrix (with the same size of square matrix A) plays the role of the number **zero**, then:

$$A.0 = 0.A = 0$$

Exercise:

Use the matrices below and perform the multiplication of matrices (where possible).

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix},\ B = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix},\ C = \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix},\ D = \begin{pmatrix} 4 & 3 \\ 1 & -1 \\ -2 & 0 \end{pmatrix}$$

Solution:

The possible matrix multiplications are given by:

- $A_{(3\times3)} \times A_{(3\times3)},\ A_{(3\times3)} \times C_{(3\times3)},\ A_{(3\times3)} \times D_{(3\times2)}$.
- $B_{(2\times3)} \times A_{(3\times3)},\ B_{(2\times3)} \times C_{(3\times3)},\ B_{(2\times3)} \times D_{(3\times2)}$.
- $C_{(3\times3)} \times C_{(3\times3)},\ C_{(3\times3)} \times A_{(3\times3)},\ C_{(3\times3)} \times D_{(3\times2)}$.
- $D_{(3\times2)} \times B_{(2\times3)}$.
- $A_{(3\times3)} \times D_{(3\times2)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & -1 \\ -2 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 1\times4+1\times1+2(-2) & 1\times3+1(-1)+2\times0 \\ 0\times4+(-1)\times1+1(-2) & 0\times3+(-1)\times(-1)+1\times0 \\ 1\times4+2\times1+1(-2) & 1\times3+2(-1)+1\times0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 4 & 1 \end{pmatrix}$$

- $A_{(3\times3)} \times A_{(3\times3)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 1 & 3 & 0 \\ 2 & 1 & 5 \end{pmatrix}$

- $A_{(3\times3)} \times C_{(3\times3)} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -1 & 7 \\ 4 & 4 & -4 \end{pmatrix}$

- $D_{(3\times2)} \times B_{(2\times3)} = \begin{pmatrix} 4 & 3 \\ 1 & -1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 11 & -11 & 13 \\ 1 & -1 & -2 \\ -4 & 4 & -2 \end{pmatrix}$

- $B_{(2\times3)} \times D_{(3\times2)} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -3 & 4 \end{pmatrix}$

- $B_{(2\times3)} \times C_{(3\times3)} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -5 & 13 \\ -3 & 0 & 14 \end{pmatrix}$

- $C_{(3\times3)} \times A_{(3\times3)} = \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 7 & 8 \\ -3 & -9 & 0 \\ 1 & 3 & 0 \end{pmatrix}$

- $C_{(3\times3)} \times A_{(3\times3)} = \begin{pmatrix} 4 & -1 & 1 \\ 1 & 2 & -4 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & -1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 13 & 13 \\ 14 & 1 \\ -13 & -7 \end{pmatrix}$

The following properties are useful and easy to prove facts. Assume that the sums and products are defined in all following cases:

Properties:

Matrices A, B and C are conformable,

- $A + B = B + A$ Commutative property.
- $\begin{cases} A(B+C) = AB + AC \\ (B+C)A = BA + CA \end{cases}$ Distributive property.
- $A(BC) = (AB)C$ Associative property .
- $AB \neq BA$ Non-Commutative property(usually).
- $A(\lambda B) = \lambda(AB)$.
- $\begin{cases} (A+B)^2 \neq A^2 + 2AB + B^2 \\ (A-B)^2 \neq A^2 - 2AB + B^2 \\ (A+B)(A-B) = A^2 - B^2 \end{cases}$ Binomials'Identities Formulas not hold (usually).
- $A^p = \underbrace{A.A...A}_{p \ factors}$ (Matrix Powers).
- $A^0 = I_n$.
- $AB = 0$ Not neccessarly imply $A = 0 \vee B = 0$.

Example 2.10

Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix}$, $C = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$ Consider the following:

$$AB = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} . \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 15 & 15 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix} . \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 9 \\ 14 & 7 \end{pmatrix}$$

Thus, $AB \neq BA$, and we see that two matrices are not the same if they are multiplied the other way around. Also consider.

$$AC = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} . \begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} -13 \\ 18 \end{pmatrix}$$

$$BC = \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix} . \begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 21 \\ 13 \end{pmatrix}$$

$$AC + BC = \begin{pmatrix} -13 \\ 18 \end{pmatrix} + \begin{pmatrix} 21 \\ 13 \end{pmatrix} = \begin{pmatrix} 8 \\ 31 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 6 & 7 \end{pmatrix}$$

$$(A + B)C = \begin{pmatrix} -5 & 2 \\ 15 & 15 \end{pmatrix} . \begin{pmatrix} -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ 31 \end{pmatrix}$$

and we notice that $(A + B)C = CA + BC$ as required.

Transpose of Matrix

The **Transpose** of A of order $m \times n$, denoted by A^t, is the $n \times m$ matrix where it is obtained from A by interchanging its rows with its columns. We say that the order of A^t is the reverse of the order of A.

That is,

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \text{ then } A^t = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

Notice that:

$$(\mathrm{A}^t)^t = A$$

Example 2.11

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 7 & -1 & \sqrt{2} \\ 0 & 9 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 & 0 \\ 2 & 0 & -1 & 9 \\ 3 & 5 & \sqrt{2} & 3 \end{pmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}^t = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix}$$

For vectors, transposition turns row vectors into column vectors and vice versa:

$$\begin{pmatrix} a & b & c \end{pmatrix}^t = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix}^t = \begin{pmatrix} a & b & c \end{pmatrix}$$

Remark:

The identity matrix, a diagonal matrix, and a scalar are equal to their transpose since the diagonal is unaffected.

Symmetric matrix

A **Symmetric Matrix** is defined as the square matrix that is equal to its transpose matrix.

$$\mathrm{A}^t = A \text{ or } a_{ij} = a_{ji}$$

A symmetric matrix A therefore satisfies the condition.

Example 2.12

The matrix below is symmetric

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 1 \end{pmatrix}$$

Here we can see that, $a_{ij} = a_{ji}$

Skew-Symmetric Matrix

Square matrix A is said to be **skow-symmetric matrix (or anti-symmetric)** if its transpose is equal the negative of the matrix itself

$$\mathrm{A}^t = -A \text{ or } a_{ij} = -a_{ji}$$

Example 2.13

The matrix below is skew-symmetric

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & -4 \\ -3 & 4 & 1 \end{pmatrix}$$

Properties of Transposed Matrix

- $(\lambda A)^t = \lambda A^t$
- $(A + B)^t = A^t + B^t$
- $\lambda(A + B)^t = \lambda A^t + \lambda B^t$
- $(AB)^t = B^t A^t$

Definition 2.2: Trace of Square Matrix

Let $A \in \mathcal{M}_n(\mathbb{K})$ (square matrix)

The **trace** of a square matrix A of order n, denoted $tr(A)$, is denoted to be the sum of elements on the main diagonal of A:

$$tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + ... + a_{nn}$$

Example 2.14

Let

$$A = \begin{pmatrix} 5 & -\frac{2}{9} & 0 \\ \sqrt{2} & -3 & 1 \\ 0 & 2 & 7 \end{pmatrix}$$

Find $tr(A), tr(A^t)$.

Solution:

$tr(A) = 5 + (-3) + 7 = 9$

$$A^t = \begin{pmatrix} 5 & \sqrt{2} & 0 \\ -\frac{2}{9} & -3 & 2 \\ 0 & 1 & 7 \end{pmatrix}$$

$tr(A^t) = 5 + (-3) + 7 = 9$

Properties:

- $tr(A + B) = tr(A) + tr(B)$.
- $tr(\lambda A) = \lambda tr(A)$.
- $tr(A^t) = tr(A)$.

Exercise:

Let

$$\begin{cases} f : \mathcal{M}_n(\mathbb{K}) & \to \mathbb{K} \\ A & \mapsto f(A) = tr(A) \end{cases}$$

Prove the following statements:

1. f is linear map.

2. $\forall A \in \mathcal{M}_n(\mathbb{K}),\ tr(A^t) = tr(A)$.

3. $tr(I_n) = n$.

4. $\forall A \in \mathcal{M}_{n\times p}(\mathbb{K}),\ \forall B \in \mathcal{M}_{p\times n}(\mathbb{K}) :\ tr(AB) \neq tr(BA)$.

2.2 The Inverse of Matrix

Definition 2.3: The Inverse of Matrix

A square matrix A is said to be **invertible** if there exists matrix B such that:

$$AB = BA = I$$

Where I is the identity matrix in this case, we call matrix B the inverse of the matrix A, which we denoted as A^{-1}.

Example 2.15

Consider $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

Evaluate AB and BA. What do you conclude ?

Solution:

$AB = BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then B is the inverse of A.

Properties of the Inverse

- If A is an invertible matrix, then its inverse must be unique.
- If the rows and the columns of the matrix are linearly independent, then the matrix is invertible (A^{-1} exists).
- A square matrix that has an inverse is called a **non-singular matrix.**
- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^{-1})^{-1} = A$.
- $(A^t)^{-1} = (A^{-1})^t$.
- $(\lambda A)^{-1} = \frac{1}{\lambda}A^{-1}$.

Example 2.16

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$$

A is an invertible matrix (see the above example).

The column vectors $(1, 1)$ and $(2, 3)$ are linearly independent

$$\forall \alpha, \beta \in \mathbb{R} : \alpha(1,1) + \beta(2.3) = (0.0) \implies \begin{cases} \alpha + 2\beta = 0 \\ \alpha + 3\beta = 0 \end{cases}$$

$$\implies \begin{cases} \alpha = -2\beta \\ \alpha = -3\beta \end{cases}$$

$$\implies \alpha = \beta = 0$$

Exercise:

Use the below square matrix A of order 3 to perform the following questions:

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

1. Calculate A^2.

2. Determine the scalars α, β, λ where $\lambda A^2 + \alpha A + \beta I = 0$ such that is the identity matrix and 0 is the null matrix.

3. From the above equality show that A is invertible and find its inverse.

Solution:

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

1. **Calculating** A^2

$$A^2 = A.A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix}$$

2. **Finding of the scalars** α, β, λ

We have $\lambda A^2 + \alpha A + \beta I = \lambda \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & 4 \\ 4 & 4 & 8 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} 8\alpha+\beta & 4\lambda+2\alpha & 4\lambda+2\alpha \\ 4\lambda+2\alpha & 8\alpha+\beta & 4\lambda+2\alpha \\ 4\lambda+2\alpha & 4\lambda+2\alpha & 8\alpha+\beta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} 8\alpha+\beta=0 \\ \\ 4\lambda+2\alpha=0 \end{cases}, \text{ then } \begin{cases} \beta=-8\lambda \\ \\ \alpha=-2\lambda \end{cases}$$

If we take $\lambda = 1$, then $\beta = -8$ and $\alpha = -2$

So that $A^2 - 2A - 8I = 0$

3. **Proving that A is invertible and finding its inverse**

From the above equality we have,

$$A^2 - 2A - 8I = 0 \implies A^2 - 2A = 8I \implies A\left[\frac{1}{8}(A-2I)\right] = I \ldots (1)$$

On the other side $A^2 - 2A - 8I = 0 \implies A^2 - 2A = 8I \implies (A-2I)A = 8I$

Then

$$\left[\frac{1}{8}(A-2I)\right] A = I \ldots (2)$$

From the equations (1) and (2), we conclude that the matrix A is invertible. So there exists a matrix

$$B = \frac{1}{8}(A-2I)$$

Satisfies

$$AB = BA = I$$

Finding A^{-1}

$$A^{-1} = B = \frac{1}{8}(A-2I)$$

$$A^{-1} = \frac{1}{8}\left(\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \frac{1}{8}\begin{pmatrix} -2 & 2 & 2 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

2.2.1 Finding Inverses With Gaussian-Jordan Method or Elementary Row Operations

For any matrix $A \in \mathcal{M}_n(K)$, we can perform **Gaussian-Jordan elimination** on the **augmented matrix**

$$\left[A \,\middle|\, I_n\right]$$

If at termination of **Gaussian elimination**, we end up with an identity matrix on the left, then the matrix on the right is the inverse of the matrix A.

$$\left[I_n \,\middle|\, A^{-1}\right]$$

If we don't end up with an identity matrix on the left after running Gaussian elimination, we know that the matrix is not invertible.

If we don't get the identity matrix, we will have a row of zero, which indicates that the rows of A are linearly dependent.

Example 2.17

A square matrix A and its inverse A^{-1} will always satisfy the following conditions:

$AA^{-1} = I$ and $A^{-1}A = I$, where I is the identity matrix.

Let $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

We want to find the values of a_{ij} such that the equation would be satisfied

$$\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}.A^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

We can write the above as an **augmented matrix**, which joins the left right numerical matrices together and hides the variable matrix, as show below

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

Now, to show the inverse matrix A^{-1} using the **Gaussian-Jordan elimination**, we have to transform the left matrix to the identity, then the right matrix become our solution. In equation form $AB = I$, we are transforming A and I simultaneously using **row operations** so that the equation becomes $I.B = A^{-1}$, where B is the resulting matrix, and we have the values for the elements in our inverse matrix. We will now do the actual computation bellow.

$$\left[A \,\middle|\, I\right] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

(R_1 is the first row andR_2 is the second row)

$$\Longrightarrow \left[A \,\middle|\, I\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{array}{c} R_1 \\ R_2 - R_1 \end{array}$$

$$\Longrightarrow \left[A \,\middle|\, I\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ R_1 \end{array}$$

Therefore, A is invertible and $B = A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

2.2.2 Transition Matrix

Definition 2.4

If B and $B^{'}$ are two bases of $\mathbb{R}^n$, with $B = \{v_1, v_2, ..., v_n\},\ B^{'} = \{w_1, w_2, ..., w_n\}$ and the coordinates of $B^{'}$ are given in B by $w_j = \sum_{i=1}^{n} p_{ij} v_i$.

The transition matrix from the basis B to the basis $B^{'}$, is given by.

$$P = (p_{ij})_{n \times n}$$

The coordinates of a vector x in $B^{'}$ are given by $[x]_{B'} = P^{-1}[x]_B$.

The coordinates of a vector x in B are given by $P[x]_{B'} = [x]_B$.

Example 2.18

The transition matrix from the **standard** basis $\{e_1 = (1, 0),\ e_2 = (0, 1)\}$ to the new basis $\{(1, 2), (3, 3)\}$ is given by $P = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}_{2 \times 2}$

2.2.3 Matrix of The Linear Transformation

Definition 2.5

Let A be an $m \times n$ matrix. The function f denoted by

$$\begin{array}{rcl} f : \mathbb{R}^n & \to & \mathbb{R}^n \\ v & \mapsto & f(v) = Av \end{array}$$

is linear map from $\mathbb{R}^n$ into $\mathbb{R}^n$

$$Av = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{pmatrix}$$

Example 2.19

$$\begin{aligned} f : \mathbb{R}^2 &\rightarrow \mathbb{R}^3 \\ v &\mapsto f(v) = Av \end{aligned}$$

$$= \begin{pmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

1. Find $f(v)$.

2. Show that f is a linear map from $\mathbb{R}^2$ into $\mathbb{R}^3$.

Solution:

1. $f(v) = Av = \begin{pmatrix} 3 & 0 \\ 2 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 2x + y \\ -x - y \end{pmatrix}$

2. $f(v + u) = A(v + u) = Av + Au = f(v) + f(u)$

   $f(\alpha v) = A(\alpha v) = \alpha(Av) = \alpha f(v)$

Example 2.20

$$\begin{aligned} f : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto f(x, y) = (3x - y, -x + 5y) \end{aligned}$$

1. Find the matrix for the linear map f.

Solution:

$f(x, y) = \begin{pmatrix} 3x - y \\ -x + 5y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. So, the matrix of f is $\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$

2.3 Determinants

Definition 2.6: Determinant

Determinant is a number associated with **a square matrix** A, denoted by $detA$ or $|A|$.

2.3.1 Evaluation of Determinants

For $A_{1\times 1}$

$$|A| = |a_{11}| = a_{11}$$

Remark:

$|a_{11}| = a_{11}$ is not a modulus.

Example 2.21

For $A = (-2)$, $detA = |-2| = -2$

For $A_{2\times 2}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} + a_{21}a_{12}$$

Example 2.22

Consider the matrix $A = \begin{pmatrix} 5 & 2 \\ 3 & -4 \end{pmatrix}$

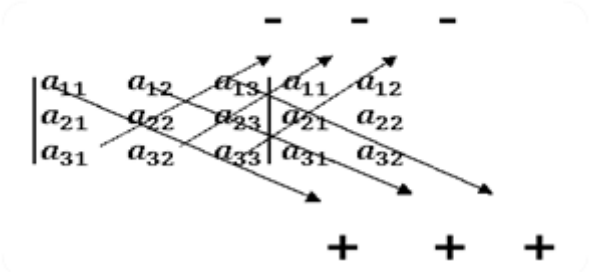
The determinant is given by $detA = \begin{vmatrix} 5 & 2 \\ 3 & -4 \end{vmatrix} = 5 \times (-4) - 3 \times 2 = -20 - 6 = -26$

For $A_{3\times 3}$

By **Sarrus Rule:** Rewrite the first tow rows.

Sarrus Rule

The rule of **Sarrus** is used for solving only for the determinant of 3×3 matrices. The method is based on **duplication of the two first columns** of the matrix to the right of its third column.Then add the products of main diagonals going from top to bottom and subtract the product of the main diagonals going from bottom to top.



$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11}$$

$$-a_{33}a_{21}a_{12}$$

Example 2.23

Let the following 3×3 matrix

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ -4 & 7 & -2 \end{vmatrix}$$

First we rewrite the first tow columns to the right of the 3×3 matrix.

$$\left|\begin{array}{ccc|cc} 1 & 2 & 1 & 1 & 2 \\ 3 & 0 & 6 & 3 & 0 \\ -4 & 7 & -2 & -4 & 7 \end{array}\right|$$

And then we add the products of the main diagonals going from top bottom and subtract the product of the main diagonals going from bottom to top.

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ -4 & 7 & -2 \end{vmatrix} \begin{matrix} 1 & 2 \\ 3 & 0 \\ -4 & 7 \end{matrix} = 1.0.(-2) + 2.6.(-4) + 1.3.7 - (-4).0.1 - 7.6.1 + 2.3.2$$

$$= -48 + 21 - 42 + 12 = -57$$

For $A_{n\times n},\ n \geq 3$

By Using **Cofactor Expansion.**

Cofactor Expansion

Submatrix of a matrix

Given a matrix $A_{n\times n}$ and take out a row i and a column j, the resulting matrix is a **submatrix** of A of order $n-1$, denoted by A_{ij}.

Example 2.24

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ -4 & 7 & -2 \end{pmatrix}$$

Find A_{23}, A_{22} and A_{31}.

Solution:

- A_{23} is obtained by deleting the row $i = 2$ and the column $j = 3$.

$$A_{23} = \begin{pmatrix} 1 & 2 \\ -4 & 7 \end{pmatrix}$$

- A_{22} is obtained by deleting the row $i = 2$ and the column $j = 2$.

$$A_{22} = \begin{pmatrix} 1 & 1 \\ -4 & -2 \end{pmatrix}$$

- A_{31} is obtained by deleting the row $i = 3$ and the column $j = 1$.

$$A_{31} = \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix}$$

Minor

The determinant $det(A_{ij})$ or $|A_{ij}|$ is called a **minor** of A.

Cofactors of a Matrix

Cofactor matrix is the minor of the element with a proper sign, denoted by C_{ij}, and given by $C_{ij} = (-1)^{i+j}|A_{ij}|$.

Definition 2.7

The Determinant of a Square Matrix A is the sum of the products **of the elements of any row (or column)** a_{ij} **and their cofactors** C_{ij}.

- **Using the** i^{th} **row**

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

Or

- **Using the** j^{th} **column**

$$|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}$$

Example 2.25

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ -4 & 7 & -2 \end{pmatrix}$$

1. Find the submatrices A_{11}, A_{12}, A_{13} and the minors $|A_{11}|, |A_{12}|, |A_{13}|$ and their cofactors C_{11}, C_{12}, C_{13}.

2. Evaluate the determinant of A using the first row.

Solution:

1. Finding the minors and cofactors for the element of the first row

   Minors for the first row $(i = 1)$: A_{11}, A_{12}, A_{13}.

$$A_{11} = \begin{vmatrix} 0 & 6 \\ 7 & -2 \end{vmatrix} = -42, \; A_{12} = \begin{vmatrix} 3 & 6 \\ -4 & -2 \end{vmatrix} = -6 + 24 = 18,$$

$$A_{13} = \begin{vmatrix} 3 & 0 \\ -4 & 7 \end{vmatrix} = 21.$$

   Cofactors for the first row $(i = 1)$: C_{11}, C_{12}, C_{13}.

$$C_{11} = (-1)^{1+1}|A_{11}| = -42.$$

$$C_{12} = (-1)^{1+2}|A_{12}| = -18.$$

$$C_{13} = (-1)^{1+3}|A_{13}| = 21.$$

2. Evaluate the determinant of A.

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ -4 & 7 & -2 \end{vmatrix} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$|A| = 1(-42) + 2(-18) + 1(21) = -57$$

Remarks:

- When calculating the determinant, you can **choose any row or column.**
- When calculating the determinant, it is better that you **choose any row or column containing many zeros.**

- Sign system for order 2 and order 3 are given by $\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$

Example 2.26

Compute the determinant of $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 6 \\ -4 & 7 & -2 \end{pmatrix}$

Solution:

We choose the second row $i = 2$ or second column $j = 2$.

Expanding along second column $j = 2$

$$|A| = a_{12}C_{12}, a_{22}C_{22}, a_{32}C_{32}$$

$$C_{i2} = (-1)^{i+2}|A_{i2}|$$

Sign system expansion

$$|A| = \begin{vmatrix} 1 & \ominus 2 & 1 \\ 3 & \oplus 0 & 6 \\ -4 & \ominus 7 & -2 \end{vmatrix} = a_{12}C_{12}, a_{22}C_{22}, a_{32}C_{32}$$

$$= -2\begin{vmatrix} 3 & 6 \\ -4 & 2 \end{vmatrix} - 7\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} - 2(-6 + 24) - 7(6 - 3)$$

$$= -36 - 21 = -57$$

Example 2.27

Compute the determinant of $A = \begin{pmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 5 \\ 2 & 2 & 2 & 4 \\ 3 & -3 & 1 & 1 \end{pmatrix}$

Solution:

Using the second row $i = 2$

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$$

$$= 5\begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & -3 & 1 \end{vmatrix} = 5\left[\begin{vmatrix} 2 & 2 \\ -3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}\right] = 5[2 + 6 + 2 - 6] = 20$$

Example 2.28

Compute the determinant of $A = \begin{pmatrix} \frac{3}{4} & -1 & 0 & 3 \\ 0 & -8 & 0 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

Solution:

By using the 4^{th} row

$$|A| = \begin{vmatrix} \frac{3}{4} & -1 & 0 & 3 \\ 0 & -8 & 0 & 5 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \frac{3}{4} & -1 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 2 \end{vmatrix} = \frac{1}{2}.(2).\frac{3}{4}.(-8) = -6$$

2.3.2 Row (Column) Operations

Consider the following notations to evaluate a determinant:

1. R_i to denote i^{th} row.
2. $R_i \leftrightarrow R_j$ to denote the interchange (or swap the rows) of i^{th} and j^{th} rows.
3. $R_i \leftrightarrow R_i + \alpha R_j$ to denote the addition of α times the elements of j^{th} row to the corresponding elements of i^{th} row.
4. αR_i to denote the multiplication of all elements of i^{th} row by α.
   Similar notations can be used to denote column operations by replacing R with C.

Properties of Determinants

- $det(A) = det(A^t)$.
- $det(A.B) = det(A)det(B)$.
- A is **invertible** if and only if $det(A) \neq 0$.
- $det(A^{-1}) = \dfrac{1}{det(A)}$.
- The determinant of a triangular matrix is the product of its diagonal elements.
- If any two rows (or columns) of a determinant are interchanged, then the value of the determinant is changed by minus sign.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & k \end{vmatrix} \quad \text{Applying } R_1 \leftrightarrow R_2$$

- If all the elements of a row (or column) are multiplied by a non-zero number α, then the value of the new determinant is α times the value of the original determinant,

$$\begin{vmatrix} \alpha a & \alpha b & \alpha c \\ d & e & f \\ g & h & k \end{vmatrix} = \alpha \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$$

- If each element of any row (or column) consists of two or more terms, then the determinant can be expressed as the sum of two or more determinants,

$$\begin{vmatrix} \alpha + a & b & c \\ \beta + d & e & f \\ \gamma + g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} \alpha & b & c \\ \beta & e & f \\ \gamma & h & k \end{vmatrix}$$

- The value of a determinant is unchanged. If any row (or column) is multiplied by a number and then added to any other row (or column).

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} = \begin{vmatrix} a + \alpha b + \beta c & b & c \\ d + \alpha e + \beta f & e & f \\ g + \alpha h + \beta k & h & k \end{vmatrix}$$

- If any two rows (or columns) of a determinant are identical, then its value is zero,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = 0$$

- If each element of a row (or column) of a determinant is zero, then its value is zero,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Example 2.29

Find the value of the following determinants:

$$\begin{vmatrix} 12 & 5 & 3 \\ 8 & 7 & 2 \\ 16 & 9 & 4 \end{vmatrix}, \quad \begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix}, \quad \begin{vmatrix} 1 & \alpha & \beta+\gamma \\ 1 & \beta & \gamma+\alpha \\ 1 & \gamma & \alpha+\beta \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 12 & 5 & 3 \\ 8 & 7 & 2 \\ 16 & 9 & 4 \end{vmatrix} = \begin{vmatrix} 4.3 & 5 & 3 \\ 4.2 & 7 & 2 \\ 4.4 & 9 & 4 \end{vmatrix} = 4 \begin{vmatrix} 3 & 5 & 3 \\ 2 & 7 & 2 \\ 4 & 9 & 4 \end{vmatrix} \text{ (Taking out 4 common from column } C_1)$$

$$= 4.0 = 0 \ (C_1 \text{ and } C_3 \text{ are identical}).$$

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} \begin{matrix} C_1 \\ C_2+2C_3 \\ C_3 \end{matrix} = \begin{vmatrix} 3 & 0 & -2 \\ -1 & 0 & 3 \\ 2 & 3 & -3 \end{vmatrix}$$

Using the second column

$$\begin{vmatrix} 3 & 4 & -2 \\ -1 & -6 & 3 \\ 2 & 9 & -3 \end{vmatrix} = (-3) \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} = (-3)(9-2) = -21$$

$$\begin{vmatrix} 1 & \alpha & \beta+\gamma \\ 1 & \beta & \alpha+\gamma \\ 1 & \gamma & \alpha+\beta \end{vmatrix} = \begin{vmatrix} 1 & \alpha & \alpha+\beta+\gamma \\ 1 & \beta & \alpha+\beta+\gamma \\ 1 & \gamma & \alpha+\beta+\gamma \end{vmatrix} \begin{matrix} C_1 \\ C_2 \\ C_3+C_2 \end{matrix}$$

$$(\alpha+\beta+\gamma) \begin{vmatrix} 1 & \alpha & 1 \\ 1 & \beta & 1 \\ 1 & \gamma & 1 \end{vmatrix} \text{ (Taking out } (\alpha+\beta+\gamma) \text{ as a common factor from column } C_3)$$

$$(\alpha+\beta+\gamma).0 = 0 (C_1 \text{ and } C_3 \text{ are identical}).$$

Example 2.30

Let $A = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{pmatrix}$, $|A| = 12$ is known.

Evaluate the determinants of the following matrices:

$$E = \begin{pmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{pmatrix}, F = \begin{pmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{pmatrix}, G = \begin{pmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{pmatrix}$$

Solution:

$\underbrace{C_2}_{E} = 3\underbrace{C_2}_{A}$ then $|E| = 3|A| = 36$.
$A \to F$ (Applying $R_2 \leftrightarrow R_3$) then $|F| = -|A| = -12$.

$A \to G$ (Applying $R_3 + 2R_1$) then $|G| = -|A| = 12$.

Example 2.31

Evaluate the determinant $\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 5 & -11 \end{vmatrix} \begin{matrix} R_1 \\ R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 - 6R_1 \end{matrix} = 1 \times 0 \times 3 \times (-11) = 0$$

2.3.3 Inverse of a Matrix Using Determinants

Adjoint Matrix

Definition 2.8: Cofactor Matrix

Cofactor Matrix (C_{ij}) of a matrix A is the square matrix of the order as A in which each element a_{ij} is replaced by its cofactor C_{ij}.

The **Adjoint matrix** of A, denoted by $\boldsymbol{adj\,A = (C_{ij})^t}$.

Example 2.32

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$$

$C_{11} = -1, C_{12} = -4, C_{21} = -3, C_{22} = +2$

The cofactor matrix of A is $(C_{ij}) = \begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}$

The Adjoint matrix of A is $adj\,A = (C_{ij})^t = \begin{pmatrix} -1 & -4 \\ -3 & 2 \end{pmatrix}^t = \begin{pmatrix} -1 & -3 \\ -4 & 2 \end{pmatrix}$

Using the Adjoint Matrix in Matrix Invertion

Definition 2.9

Let A be a square matrix with $|A| \neq 0$. A is **invertible** with

$$\boldsymbol{A^{-1} = \frac{1}{|A|}\, adj\, A}$$

Example 2.33

Use a determinant to find out which of the following matrices are invertible.

$$A = \begin{pmatrix} 5 & 1 \\ 10 & 2 \end{pmatrix},\ B = \begin{pmatrix} 3 & 2 \\ 4 & 0 \end{pmatrix},\ C = \begin{pmatrix} 1 & 3 & 5 \\ 4 & 0 & 2 \\ -2 & -6 & -10 \end{pmatrix},\ D = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & 8 & 0 \end{pmatrix}$$

Solution:

$|A| = 0$, A is not invertible. The inverse does not exist.

$|B| = -8 \neq 0$, B is invertible.

$|C| = 0$, C is not invertible. The inverse does not exist.

$|D| = 2 \neq 0$, D is invertible.

Example 2.34

Use the formula for the inverse of a matrix to compute the inverse of the following matrices.

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$$

$|A| = \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} = -2 - 12 = -14 \neq 0$, then A is invertible.

$$A^{-1} = \frac{1}{|A|} adj A$$

Such that $(adjA) = (C_{ij})^t$ and $C_{ij} = (-1)^{i+j}|A_{ij}|$

$$A^{-1} = \frac{\begin{pmatrix} -1 & -3 \\ -4 & 2 \end{pmatrix}}{-14} = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{2}{7} & \frac{-1}{7} \end{pmatrix}$$

Let's examine the outcome when we multiply the matrix by its inverse.

$$A.A^{-1} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} . \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{2}{7} & \frac{-1}{7} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1}.A = \begin{pmatrix} \frac{1}{14} & \frac{3}{14} \\ \frac{2}{7} & \frac{-1}{7} \end{pmatrix} . \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix}$$

$$B = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 2 & 0 & -1 \end{vmatrix} \begin{matrix} C_1 \\ C_1 + C_2 \\ C_3 \end{matrix}$$

Using the second column $B = 3 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -3 \neq 0$, so the inverse of B exists.

$$B^{-1} = \frac{1}{|B|}(adjB)$$

Such that $(adjB) = (C_{ij})^t$ and $C_{ij} = (-1)^{i+j}|B_{ij}|$.

We give the matrix of cofactors and the Adjoint matrix of the matrix.

$$(C_{ij}) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

$$(C_{ij}) = \begin{pmatrix} +\begin{vmatrix} 2 & 1 \\ -2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$$

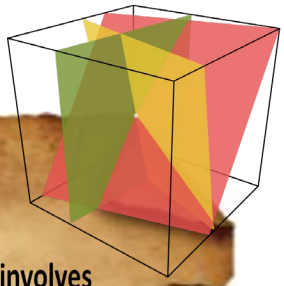
$$(C_{ij}) = \begin{pmatrix} 0 & 3 & -6 \\ -1 & -1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$$

$$(adjB) = (C_{ij})^t = \begin{pmatrix} 0 & -1 & -1 \\ 3 & -1 & -1 \\ -6 & 0 & 3 \end{pmatrix} \implies B^{-1} = \frac{-1}{3}\begin{pmatrix} 0 & -1 & -1 \\ 3 & -1 & -1 \\ -6 & 0 & 3 \end{pmatrix}$$

$$B.B^{-1} = \frac{-1}{3}\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & -1 \\ 3 & -1 & -1 \\ -6 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$B^{-1}.B = \frac{-1}{3}\begin{pmatrix} 0 & -1 & -1 \\ 3 & -1 & -1 \\ -6 & 0 & 3 \end{pmatrix}.\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ 2 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Systems of Linear Equations



Linear algebra is the study of a mathematical structure called a vector space. It involves analyzing linear sets of equations and their transformations, as well as investigating properties of finite-dimensional vector spaces and linear mappings. The history of linear algebra can be traced back thousands of years, with early civilizations such as Babylon and ancient China demonstrating the ability to solve systems of linear equations. However, significant advancements in the field came in the late 17th century with the study of determinants by mathematicians like Leibnitz and the development of techniques such as Lagrange multipliers by Lagrange. Cramer also made contributions to solving systems of linear equations based on determinants. Despite its ancient origins, linear algebra has gained more relevance over time, especially with the emergence of calculus.

The history of linear algebra is characterized by important contributions from mathematicians such as Euler, Gauss, Cayley, and others. Euler recognized the need for conditions on unknown variables to find solutions to systems of equations. Gauss introduced Gaussian elimination, a procedure for solving systems of linear equations. Cayley's work on matrix multiplication and the use of matrices in algebra further advanced the field. The development of technology, particularly computers, greatly accelerated the practical applications of linear algebra. It is widely used in various fields, including physics, data analysis, and solving differential equations. Despite technological advancements, the fundamentals of linear algebra remain unchanged, making it a foundational subject in mathematics.

Extracted from “A Brief History of Linear Algebra” Jeff Christensen April 2012 Final Project Math 2270 Grant Gustafson University of Utah.

Chapter 3

Systems of Linear Equations

3.1 Solving Linear Systems by Matrix Methods

A system of linear equations is common and important for servery problems in science, engineering and mathematics. For example in determining the forces in a large framework or finding currents in a complicated electrical circuit.

A system of linear equations can be expressed in **matrix from** and we can solve using **Cramer's Rule , Inverse Matrix** and **Gaussian Elimination**

- **Cramer's Rule** and **Inverse Matrix Method** are useful for solving simple linear systems of n equations in n unknown variables.
- The method of **Gauss Elimination** is used to solve large and complex systems of m linear equations in n unknown variables.

Definition 3.1

A **system of** m **linear equations** in n **unknown.**

$$(S)\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

The system (S) may be expressed in the **matrix form.**

$$AX = B$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \ddots & \ddots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ **Coefficients matrix.**

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$ **Vector of unknowns.**

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$ **Vector of constants.**

Example 3.1

$$(S)\begin{cases} -x - 2y + 3z + t = 1 \\ x - y + z - 2t = 2 \\ 2x - 8y - 3z + t = -1 \end{cases}$$

- (S) is system of 3 equations with 4 unknowns.
- The **matrix form** is given by $AX = B$, such that:

$$A = \begin{pmatrix} -1 & -2 & 3 & 1 \\ 1 & -1 & 1 & -2 \\ 2 & -8 & -3 & 1 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Notice:

If $AX = 0$ then the system is called **homogeneous** linear system of equations.

- If $A \neq 0$, then the system has a **trivial solution:**

$$x_1 = x_2 = \cdots = x_n = 0$$

- If $|A| = 0$, then the system has **infinite solutions.**

3.1.1 Solution by Cramer's Rule

Cramer's Rule

Used to solve system of linear equations with determinants.

- The number of equations is equal to the number of unknowns.
- If $|A| \neq 0$, then the system is consistent and has a **unique solution**.

$$x_1 = \frac{|A_1|}{|A|}, x_2 = \frac{|A_2|}{|A|}, \cdots, x_n = \frac{|A_n|}{|A|}$$

- A : Coefficients matrix.
- A_n : Coefficients matrix with **column n replaced with constants.**

If $|A| = 0$ this method of solution can not be used.

Example 3.2

Use Cramer's Rule to solve the following linear systems:

1. $\begin{cases} 2x + y = 1 \\ -x + 3y = 4 \end{cases}$

2. $\begin{cases} 5u - v + 4w = 5 \\ 2u + 3v + 5w = 2 \\ 5u - 2v + 6w = -1 \end{cases}$

Solution:

1. $\begin{cases} 2x + y = 1 \\ -x + 3y = 4 \end{cases}$

$$\textbf{Matrix form} \qquad AX = B \iff \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

(a) The **number of equations** is equal to the **unknowns.**

(b) $|A| = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - (-1) = 7 \neq 0 \implies$ The system **has a unique solution.**

$$|A_x| = \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 3 - 4 = -1$$

$$|A_y| = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7$$

By Carmer's rule

$$x = \frac{|A_x|}{|A|} = \frac{-1}{7}$$

$$y = \frac{|A_y|}{|A|} = \frac{7}{7} = 1$$

The set of solutions $\left\{\left(\frac{-1}{7}, 1\right)\right\}$

2. $$\begin{cases} 5u - v + 4w = 5 \\ 2u + 3v + 5w = 2 \\ 5u - 2v + 6w = -1 \end{cases}$$

$$AX = B \iff \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

(a) The **number of equations** is equal to the **unknowns.**

(b)

$$|A| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{vmatrix} = \begin{vmatrix} C_1 + 5C_2 & C_2 & C_3 + 4C_2 \\ 0 & -1 & 0 \\ 17 & 3 & 17 \\ -5 & -2 & -2 \end{vmatrix} = \begin{vmatrix} 17 & 17 \\ -5 & -2 \end{vmatrix} = 17 \begin{vmatrix} 1 & 1 \\ -5 & -2 \end{vmatrix}$$

$$= 17(-2 + 5) = 17 \times 3 = 51$$

$|A| = 51 \neq 0 \implies$ The system **has a unique solution.**

$$|A_u| = \begin{vmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ -1 & -2 & 6 \end{vmatrix} = \begin{vmatrix} C_1 + 5C_2 & C_2 & C_3 + 4C_2 \\ 0 & -1 & 0 \\ 17 & 3 & 17 \\ -11 & -2 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 17 & 17 \\ -11 & -2 \end{vmatrix} = +17 \times \begin{vmatrix} 1 & 1 \\ -11 & -2 \end{vmatrix} = 17 \times 9 = 153$$

$|A_u| = 153$

$$|A_v| = \begin{vmatrix} 5 & 5 & 4 \\ 2 & 2 & 5 \\ 5 & -1 & 6 \end{vmatrix} = \begin{vmatrix} C_1 - C_2 & C_2 & C_3 \\ 0 & 5 & 4 \\ 0 & 2 & 5 \\ 6 & -1 & 6 \end{vmatrix} = 6 \begin{vmatrix} 5 & 4 \\ 2 & 5 \end{vmatrix} = 6 \times 17 = 102$$

$|A_v| = 102$

$$|A_w| = \begin{vmatrix} 5 & -1 & 5 \\ 2 & 3 & 2 \\ 5 & -2 & -1 \end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3 - C_1 \\ 5 & -1 & 0 \\ 2 & 3 & 0 \\ 5 & -2 & -6 \end{vmatrix}$$

$$-6 \begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} = -6(15 + 2) = -6 \times 17 = -102$$

$|A_w| = -102$

By Carmer's rule

$u = \dfrac{|A_u|}{|A|} = \dfrac{153}{51} = 3.$

$v = \dfrac{|A_v|}{|A|} = \dfrac{102}{51} = 2.$

$w = \dfrac{|A_w|}{|A|} = \dfrac{-102}{51} = -2.$

The set of solutions $\{(3, 2, -2)\}$

3.1.2 Solution by Inverse Matrix Method

Inverse Matrix

- The number of equations is equal to the number of unknowns.
- Used an inverse to solve system of equations.
- Write system in matrix form $AX = B$.

$$A^{-1}AX = A^{-1}B \implies IX = A^{-1}B \implies X = A^{-1}B$$

- Solve by multiplying the inverse of the coefficients matrix with the vector constants.

Example 3.3

Use the inverse matrix method to solve the previous system:

$$\begin{cases} 5u - v + 4w = 5 \\ 2u + 3v + 5w = 2 \\ 5u - 2v + 6w = -1 \end{cases}$$

Matrix form

$$AX = B \iff \begin{pmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$

1. The **number of equations** is equal to the **unknowns.**
2. $|A| = 51 \neq 0 \implies A^{-1}$ exists.

The system **has a unique solution** $X = A^{-1}B$.

$$A^{-1} = \frac{adjA}{|A|} = \frac{(C_{ij})^t}{|A|}$$

Cofactors matrix

$$(C_{ij}) = \begin{pmatrix} +\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 5 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} & +\begin{vmatrix} 5 & 4 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 5 & -1 \\ 5 & -2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 4 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 5 & 4 \\ 2 & 5 \end{vmatrix} & +\begin{vmatrix} 5 & -1 \\ 2 & 3 \end{vmatrix} \end{pmatrix}$$

$$(C_{ij}) = \begin{pmatrix} 28 & 13 & -19 \\ -2 & 10 & 5 \\ -17 & -17 & 17 \end{pmatrix}$$

$$(C_{ij})^t = \begin{pmatrix} 28 & -2 & -17 \\ -13 & 10 & -17 \\ -19 & 5 & 17 \end{pmatrix}$$

$$A^{-1} = \frac{1}{51}\begin{pmatrix} 28 & -2 & -17 \\ -13 & 10 & -17 \\ -19 & 5 & 17 \end{pmatrix} = \begin{pmatrix} \frac{28}{51} & -\frac{2}{51} & -\frac{1}{3} \\ -\frac{13}{51} & \frac{10}{51} & \frac{1}{3} \\ -\frac{19}{51} & \frac{55}{51} & \frac{1}{3} \end{pmatrix}$$

$$X = A^{-1}B = \begin{pmatrix} \frac{28}{51} & -\frac{2}{51} & -\frac{1}{3} \\ -\frac{13}{51} & \frac{10}{51} & \frac{1}{3} \\ -\frac{19}{51} & \frac{55}{51} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

The set of solutions $\{(3, 2, -2)\}$

3.1.3 Solution by Gauss Elimination

Elementary Row Operations

There are three possible row operations (or row reduction):

- **Interchange row i with row j (or swap)** $R_i \leftrightarrow R_j$.
- **Multiplying any row** $\alpha R_i \to R_i$ **(or scale)**.
- **Adding a multiple of one row to another** $\alpha R_i + R_j \to R_j$.
  (or linear combination of rows).

Notice:

R_i **pivot row** is a non-zero row whose **element leading must to be non-zero.**

Rank of a Matrix

The following definitions are equivalent:

Definition 3.2

The **rank of a matrix** $A \in \mathcal{M}_{m\times n}(\mathbb{K})$, written $r(A)$, is defined as

- The maximum number of linearly independent row vectors(or column vectors).
- The number of pivots (or non-zero rows) of its reduced row form obtained from the elementary row operations.

Example 3.4

What is the rank of the following matrix ?

$$\begin{pmatrix} 2 & -5 & 2 & -1 & 0 \\ -4 & 2 & 3 & 1 & 2 \\ -2 & -3 & 5 & 0 & 2 \end{pmatrix}$$

Solution:

$$A = \begin{pmatrix} 2 & -5 & 2 & -1 & 0 \\ -4 & 2 & 3 & 1 & 2 \\ -2 & -3 & 5 & 0 & 2 \end{pmatrix} \approx \begin{pmatrix} 2 & -5 & 2 & -1 & 0 \\ 0 & -8 & 7 & -1 & 2 \\ 0 & -8 & 7 & -1 & 2 \end{pmatrix} \begin{matrix} R_1(\text{Pivot row}) \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix}$$

$$A \approx \begin{pmatrix} 2 & -5 & 2 & -1 & 0 \\ 0 & -8 & 7 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1 \\ R_2(\text{Pivot row}) \\ R_3 - R_2 \end{matrix}$$

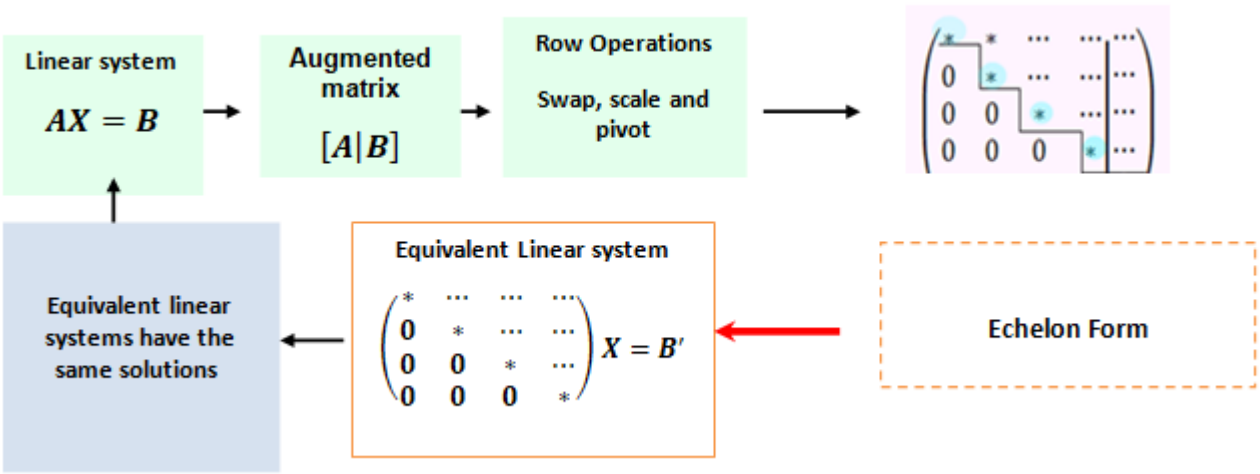
The number of pivots (or non-zero rows) in the reduced row of A is equal to 2. Then the rank $r(A) = 2$

Gaussian elimination

The **Gaussian elimination** (or known as **row echelon form**) is based on three stages:

1. Give the matrix form $AX = B$ and the **augmented matrix** $[A|B]$.
2. Perform **rows operations** on $[A|B]$ to bring the **matrix** A to **upper triangular form.**
3. Solve the equivalent final system by **back-substitution.**

Illustration: Gauss method (or Row Echelon Form REF)



Determining Consistency

Given the linear system $AX = B$ and the **augmented matrix** $[A|B]$.

- If $r(A) = r[A|B] = n$ the number of unknowns, then the system is **consistent** and has a **unique solution**.
- If $r(A) = r[A|B] < n$ the number of unknowns, then the system is **consistent** and has **infinitely many solutions.** So the system **would have a free variables (or free parameter).**

$$\textbf{Number of free parameters} = n - r(A)$$

- If $r(A) \neq r[A|B]$, then the system is **inconsistent** and **has no solution.**

Illustration

$$\left(\begin{array}{cccc|c} \star & \cdots & \cdots & \cdots & \cdots \\ 0 & \star & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \star & \star \end{array}\right) \Longrightarrow \textbf{Exactly one solution.}$$

$$\left(\begin{array}{cccc|c} \star & \cdots & \cdots & \cdots & \cdots \\ 0 & \star & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow \textbf{Infinitely many solutions.}$$

$$\left(\begin{array}{cccc|c} \star & \cdots & \cdots & \cdots & \cdots \\ 0 & \star & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \star \end{array}\right) \Longrightarrow \textbf{No solution.}$$

Example 3.5

Solve the linear system by Gauss elimination:

$$\begin{cases} x - 6y - 4z = -5 \\ 2x - 10y - 9z = -4 \\ -x + 6y + 5z = 3 \end{cases}$$

Solution:

1. **The matrix form of the system is:**

$$\begin{pmatrix} 1 & -6 & -4 \\ 2 & -10 & -9 \\ -1 & 6 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}$$

2. **The augmented matrix of system is**

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

3. **Put $[A|B]$ in a row echelon form (REF):**

$$[A|B] \approx \left[\begin{array}{ccc|c} 1 & -6 & -4 & -5 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} R_1(\text{Pivot row}) \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array}$$

4. **The equivalent triangular system is given by:**

$$\begin{pmatrix} 1 & -6 & -4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ 6 \\ -2 \end{pmatrix} \iff \begin{cases} x - 6y - 4z = -5 \\ 2y - z = 6 \\ z = -2 \end{cases}$$

5. **Back-substitution:**

   From the last equation we obtain that $z = -2$. Substituting $z = -2$ into the second equation we obtain that $y = 2$. Substituting $z = -2$ and $y = 2$ into the first equation we obtain that $x = -1$.

   The set of solution $\{(-1, 2, -2)\}$

Example 3.6

Solve the linear system by Gauss elimination:

$$\begin{cases} 2x_1 + 4x_2 + 6x_3 = 8 \\ x_1 + 2x_2 + 4x_3 = 8 \\ 3x_1 + 6x_2 + 9x_3 = 12 \end{cases}$$

Solution:

1. **The matrix form of the system is:**

$$\begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 4 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ 12 \end{pmatrix}$$

2. **The augmented matrix of system is:**

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

3. **Converting $[A|B]$ to the Row Echelon Form (REF):**

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{array}\right] R_1 \leftrightarrow R_2$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{array}\right] \begin{matrix} R_1(\text{Pivot row}) \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} R_1 \\ R_2(\text{Pivot row}) \\ R_3 - \frac{3}{2}R_2 \end{matrix}$$

$r(A) = r[A|B] = 2$

n= **Number of unknown variables =3**

Since $r(A) = r[A|B] < n$, then the system is **consistent** and has **infinitely many solutions.**

This means that the solution set will contain $3-2=1$ one parameter **or free parameter.**

4. **The equivalent triangular system is given by:**

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 0 \end{pmatrix} \iff \begin{cases} x_1 + 2x_2 + 4x_3 = 8 \\ -2x_3 = -8 \end{cases}$$

From the second equation we obtain $x_1 + 2x_2 = -8$

We chose one of the variables x_1 or x_2 to be a parameter, say t, and solve for the remaining variable. If we consider $x_2 = t$ then

$$x_1 = -2t - 8$$

We give the solution set as $\{(x_1, x_2, x_3) = (-2t-8, t, 4),\ t \in \mathbb{R}\}$.

Example 3.7

Solve, if possible, the system represented by the augmented matrix:

$$\left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{array}\right]$$

Solution:

$$\left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 4 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{array}\right] \approx \left[\begin{array}{cccc} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \end{array}\right] \begin{array}{c} R_1(\text{Pivot row}) \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 3 & -1 & -3 \\ 0 & 2 & 2 & 1 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 1 & -2 \\ 0 & 2 & 2 & 1 \end{array}\right] \quad R_3 \leftrightarrow R_4$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right] \quad R_3 - \frac{2}{3}R_2$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\approx \left[\begin{array}{cccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & -\frac{13}{4} \end{array}\right] \quad R_4 - \frac{1}{3}R_4$$

$r(A) = 3$ and $[A|B] = 4$, then $r(A) \neq r[A|B]$

The last row equivalent to the equation $0x_1 + 0x_2 + 0x_3 = -\dfrac{13}{4}$ (contradiction).

The system has no solution. The set solution is empty set.

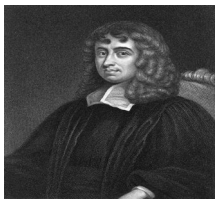
History of Integration

Several mathematicians made significant contributions to the development of integration throughout history. Here are some of them:

- **Archimedes**: Developed the method of exhaustion, a precursor to integration, to determine areas and volumes of geometric shapes.



- **Isaac Barrow**: Introduced the concept of integration and worked on the fundamental theorem of calculus.



- **Pierre de Fermat:** Developed techniques for finding areas and tangents to curves, laying the foundation for integral calculus.



- **John Wallis:** Contributed to the development of the integral symbol and made advancements in the understanding of infinite series.



- **Isaac Newton:** Developed a comprehensive framework for calculus, including the fundamental theorem of calculus, and made significant contributions to integral calculus.



- **Gottfried Wilhelm Leibniz:** Independently developed calculus, including integral calculus, and introduced the notation and rules that are still used today.

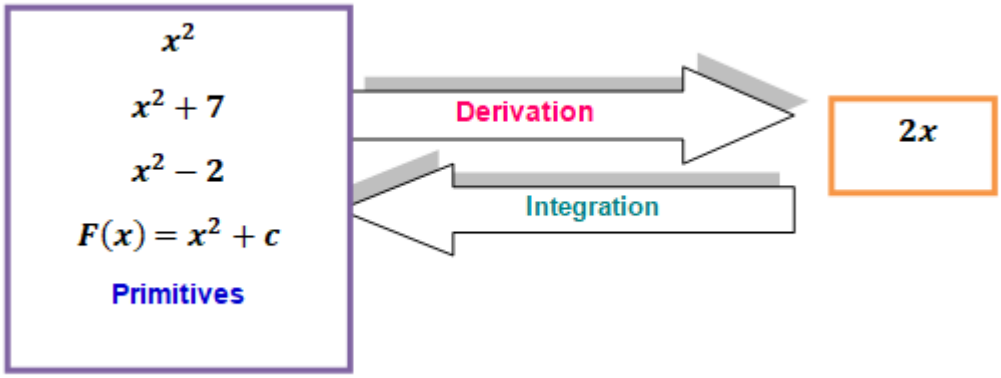


Chapter 4

Primitives and Integrations

4.1 Primitive Functions

The process of **integration** reverses the operation of **differentiation.**



The **reverse of a derivative** is a function F whose derivative is equal to the original function f.

$$f \xrightarrow{\frac{d}{dx}} f'$$

$$f' \xrightarrow{\int} f$$

Definition 4.1

A function F is called a **primitive** or a **reverse of a derivative**(or **antiderivitive**) of a continuous f on an interval I if

$$\forall x \in I : \ F^{'}(x) = f(x)$$

Theorem 4.1

If F is a **reverse of a derivative** of f on an interval I, then the most general reverse of a derivative of f on given interval I is $F(x) + c$ where c is an arbitrary constant.

4.1.1 Table of Elementary Primitives

Function f	**Primitive** F
$x^n, \ n \neq -1$	$\frac{1}{n+1}x^{n+1} + c$
$\frac{1}{x}$	$\ln \lvert x \rvert + c$
e^x	$e^x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$\frac{1}{\cos^2 x}$	$\tan x + c$
$\frac{1}{x^2+1}$	$\arctan x + c$

$\frac{1}{x^2+a^2}$	$\frac{1}{a}\arctan(\frac{x}{a})+c$
$\frac{1}{\sqrt{x^2+a^2}}$	$\ln\left\|x+\sqrt{x^2+a^2}\right\|+c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\left(\frac{x}{a}\right)+c$

Exercise:

A particle is moving with data. Find the position of particle.

Acceleration $a(t) = 10 + 3t - 3t^2$.

Initial velocity $v(0) = 2$.

Initial position $x(0) = 5$.

Solution:

$v'(t) = a(t) = 10 + 3t - 3t^2$

$v(t)$ is the primitive of $a(t)$

Then

$$v(t) = 10t + \frac{3}{2}t^2 - t^3 + c$$

$v(0) = 2 \implies c = 2$

$x(t)$ is the primitive of $v(t) = 10t + \frac{3}{2}t^2 - t^3 + 2$

$x(t) = \frac{10}{2}t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + 2t + c$

$x(0) = 5 \implies c = 5$

Thus

$$x(t) = 5t^2 + \frac{t^3}{2} - \frac{1}{4}t^4 + 2t + 5$$

Notation:

The primitive of a function f is called **indefinite integral** and is denoted by:

$$\int f(x)dx = F(x) + c$$

Where **c** is an arbitrary constant.

4.2 Indefinite Integral

4.2.1 Basic Integration Rules

For constants $\alpha, \beta \in \mathbb{R}$ and integrable functions f and g. We have

1. $\int(\alpha f(x) + \beta g(x))dx = \alpha \int f(x)dx + \beta \int g(x)dx + c.$
2. $\int F'(x)dx = F(x) + c.$
3. $\dfrac{d(\int f(x)dx)}{dx} = f(x).$

Exercise:

Evaluate the following integrals

1. $\int e^{-10x}dx.$
2. $\int \dfrac{x^3+2}{x}dx.$
3. $\int \left(\sqrt[3]{x^5} + \dfrac{8}{x^7} + \dfrac{\cos x}{\sqrt{2}} + \dfrac{3}{x^2+1}\right)dx.$
4. $\int \left(\dfrac{3}{x^2+1} + \dfrac{1}{\sqrt{25-x^2}}\right)dx.$

Solution:

1. $\int e^{-10x}dx = \dfrac{-1}{10}e^{-10x} + c$
2. $\int \dfrac{x^3+2}{x}dx = \int \left(\dfrac{x^3}{x} + \dfrac{2}{x}dx\right) = \int x^2dx + 2\int \dfrac{1}{x} = \dfrac{x^3}{3} + 2\ln|x| + c$

3. $\int \left(\sqrt[3]{x^5}+\frac{8}{x^7}+\frac{\cos x}{\sqrt{2}}+\frac{3}{x^2+1}\right)dx = \int x^{\frac{5}{3}}dx+8\int x^{-7}dx+\frac{\sqrt{2}}{2}\int \cos x dx+3\int \frac{1}{x^2+1}dx$

$$= \frac{3}{8}x^{\frac{8}{3}} - \frac{4}{3}x^{-6} + \frac{\sqrt{2}}{2}\sin x + 3\arctan x + c$$

4. $\int \left(\frac{3}{x^2+1} + \frac{1}{\sqrt{25-x^2}}\right)dx = \int \frac{3}{x^2+1}dx + \int \frac{1}{\sqrt{25-x^2}}dx$

$$= 3\int \frac{1}{x^2+1}dx + \int \frac{1}{\sqrt{25-x^2}}dx$$

$$= 3\arctan(x) + \arcsin\left(\frac{x}{5}\right) + c$$

4.2.2 Techniques of Integration

Primitive functions (on the intervals where they are defined) can not be expressed using elementary functions. Using the following techniques, we can reduce the task of finding the primitives.

Integration by Parts

To calculate the derivative of the product of two functions, we have:

$$(f(x).g(x))' = f'(x).g(x) + f(x).g(x)'$$

By inverting it, we will get the following important result for primitive functions.

Definition 4.2

Suppose f', g' are both continuous on a given interval I.To integrate some products we can use the formula:

$$\int f(x).g'(x)dx = f(x)g(x) - \int f'(x).g(x)dx$$

$$\int (f(x).g(x))'dx = \int f'(x).g(x)dx + \int f(x).g(x)'dx$$

$$\int f(x).g(x)^{'}dx = \int (f(x).g(x))^{'}dx - \int f^{'}(x).g(x)dx$$

$$\int f(x).g(x)^{'}dx = f(x).g(x) - \int f^{'}(x).g(x)dx$$

Remark: We can represent this process schematically as follows:

$$\int f(x).g(x)^{'}dx = f(x) \times g(x) - \int f^{'}(x) \times g(x)dx$$

Notice:

1. Sometimes we have to calculate the integral by parts several times.

2. Choosing the two functions f and $g^{'}$ is not arbitrary.

Care 1:

Consider the following integration

$$\int \mathcal{P}(x).e^{x}dx \ , \ \int \mathcal{P}(x).\cos xdx \ , \ \int \mathcal{P}(x).\sin xdx$$

$\mathcal{P}(x)$ is a polynomial of order n. We have to integrate by part n times by taking:

$$f(x) = \mathcal{P}(x) \text{ and } g^{'}(x) = e^{x} \text{ or } \cos x \text{ or } \sin x$$

Care 2:

Consider the following integration:

$$\int \mathcal{P}(x).\ln xdx \ , \ \int \mathcal{P}(x).\arctan xdx \ , \ \int \mathcal{P}(x).\arcsin xdx$$

$\mathcal{P}(x)$ is a polynomial of order n. We have to integrate by part only once by taking:

$$f(x) = \ln x \text{ or } \arctan x \text{ or } \arcsin x \text{ and } g^{'}(x) = \mathcal{P}(x)$$

Example 4.1

Find the integral:

$$\int x^{2}.e^{x}dx$$

Solution:

We integrate 2 times:

$$\int x^2.e^x dx = x^2.e^x - \int 2xe^x dx = x^2.e^x - 2[xe^x - e^x] + c$$

$$= e^x[x^2 - 2x + 2] + c$$

Example 4.2

Evaluate the integral:

$$\int (x^2 + 1)\cos x dx$$

Solution:

We integrate 2 times:

$$\int (x^2 + 1)\cos x dx = (x^2 + 1)\sin x - \int 2x \sin x dx$$

$$= (x^2 + 1)\sin x + 2(x\cos x + \int 1.\cos x dx)$$

$$= (x^2 + 1)\sin x + 2x\cos x + \sin x + c$$

$$= (x^2 + 2)\sin x + +2x\cos x + c$$

Example 4.3

Evaluate the integral:

$$\int x^n \ln x dx \ , \ n \neq -1$$

Solution:

We integrate only once:

$$\int x^n \ln x dx = x^{n+1}\left(\frac{\ln x}{n+1} - \frac{1}{(n+1)^2} + c\right)$$

Example 4.4

Evaluate the integral:

$$\int x \arctan x dx$$

Solution:

We integrate only once:

$$\int x \arctan x dx = \frac{1}{2}x^2 \arctan x - \frac{x}{2} - \frac{1}{2}\arctan x + c$$

Integration by Substitution

To calculate the indefinite integral

$$\int f(\mathcal{U}(x)).dx$$

We have to perform.

- Suppose $\mathcal{U}(x) = t$.
- Then $x = \mathcal{V}(t)$ that implies
$$\frac{dx}{dt} = \mathcal{V}^{'}(t)$$
- One obtains
$$dx = \mathcal{V}^{'}(t)dt$$
- We substitute x and dx in the integral in terms of t and dt.
- We get
$$\int f(t)\mathcal{V}^{'}(t)dt$$
- Change the solution back to its original form by replacing t by $\mathcal{U}(x)$.
- Check the answer by taking the derivative of the solution.

In the following examples we will solve integrals using the above substitution method.

Example 4.5

Evaluate the integral:

$$\int \sin(2x - 6)dx$$

Solution:

In this case, we should use the following as our substitution.

$$t = 2x - 6$$

$$t = 2x - 6 \implies dt = 2dx \implies dx = \frac{dt}{2}$$

Doing the substitution and evaluating the integral gives:

$$\int \sin(2x-6)dx = \int \sin t \frac{dt}{2} = \frac{1}{2}\cos t + c$$

Then

$$\int \sin(2x-6)dx = \frac{1}{2}\cos(2x-6) + c$$

Example 4.6

Evaluate the integral:

$$\int \sqrt{\sin x}\cos x dx$$

Solution:

Here an obvious substitution is:

$$t = \sin x$$

$$t = \sin x \implies dt = \cos x dx$$

Doing the substitution and evaluating the integral gives:

$$\int \sqrt{\sin x}\cos x dx = \int \sqrt{t} dt = \frac{2}{3}t^{\frac{3}{2}} + c = \frac{2}{3}(\sin x)\frac{3}{2} + c$$

Example 4.7

Find the integral:

$$\int \frac{3}{5x-1}dx$$

Solution:

In the case, we should use following as our substitution

$$t = 5x - 1$$

$$t = 5x - 1 \implies dt = 5dx$$

$$\implies dx = \frac{dt}{5}$$

Doing the substitution and evaluating the integral gives:

$$\int \frac{3}{5x-1}dx = \int \frac{3}{t} \times \frac{1}{5}dt = \frac{3}{5}\int \frac{dt}{t}$$

$$= \frac{3}{5}\ln|t| + c$$

$$= \frac{3}{5}\ln|5x-1| + c$$

Integration of Rational Functions

Relatively wide classes of functions to which primitive functions can be computed are rational functions, which are fractions of polynomials. Let's give simple examples.

$$I_1(x) = \int \frac{dx}{ax^2 + bx + c}$$

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

Such that $\Delta = b^2 - 4ac$

Case 1: $\Delta = 0$

$$I_1(x) = \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2}$$

In this case, we should use the following as our substitution

$$t = x + \frac{b}{2a}$$

Thus $dt = dx$

Doing the substitution and evaluating the integral gives:

$$I_1(t) = \frac{1}{a}\int \frac{dt}{t^2} = \frac{-1}{at} + c$$

Then

$$I_1(t) = \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2} = \frac{-1}{a\left(x + \frac{b}{2a}\right)} + c$$

Case 1: $\Delta = 0$

$$I_1(t) = \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2} = \frac{-1}{a\left(x + \frac{b}{2a}\right)} + c$$

Case 2: $\Delta > 0$

$$I_1(x) = \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\Delta}{4a^2}\right)}$$

In this case, we should use the following as our substitutions

$$t = x + \frac{b}{2a} \text{ and } B^2 = \frac{\Delta}{4a^2}$$

Thus $dt = dx$

Doing the substitution and evaluating the integral gives:

$$I_1(t) = \frac{1}{a}\int \frac{dt}{t^2 - B^2}$$

$$I_1(t) = \frac{1}{a}\int \frac{dt}{t^2 - B^2} = \frac{1}{2aB}\ln\left[\frac{t - B}{t + B}\right] + c$$

$$= \frac{1}{2a\left(\frac{\Delta}{4a^2}\right)} \ln \left| \frac{x + \frac{b}{2a} - \frac{\sqrt{\Delta}}{2a}}{x + \frac{b}{2a} + \frac{\Delta}{4a^2}} \right| + c$$

$$= \frac{1}{2a\left(\frac{\sqrt{\Delta}}{2a}\right)} \ln \left| \frac{x - \left(\frac{-b}{2a} + \frac{\sqrt{\Delta}}{2a}\right)}{x - \left(\frac{-b}{2a} - \frac{\sqrt{\Delta}}{2a}\right)} \right| + c$$

$$= \frac{1}{a(x_1 - x_2)} \ln \left| \frac{x - x_1}{x - x_2} \right| + c$$

Such that x_1 and x_2 are the roots of the denominator $ax^2 + bx + c$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore

$$I_1(x) = \frac{1}{a(x_1 - x_2)} \ln \left| \frac{x - x_1}{x - x_2} \right| + c$$

Case 2: $\Delta > 0$

$$I_1(x) = \int \frac{dx}{ax^2 + bx + c} = \frac{1}{a(x_1 - x_2)} \ln \left| \frac{x - x_1}{x - x_2} \right| + c$$

Case 3: $\Delta < 0$

$$I_1(x) = \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\Delta}{4a^2}\right)}$$

$$I_1(x) = \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \left(\frac{\sqrt{-\Delta}}{2a}\right)}$$

In this case, we should use the following as our substitutions

$$t = x + \frac{b}{2a} \text{ and } B = \frac{\sqrt{-\Delta}}{2a}$$

Thus $dt = dx$

Doing the substitution and evaluating the integral gives:

$$I_1(t) = \frac{1}{a} \int \frac{dt}{t^2 + B^2}$$

$$I_1(t) = \frac{1}{a}\int \frac{dt}{t^2 + B^2} = \frac{1}{aB}\arctan\left(\frac{t}{B}\right) + c$$

Therefore

$$I_1(x) = \frac{2}{\sqrt{-\Delta}}\arctan\left(\frac{x + \frac{b}{2a}}{\frac{\sqrt{-\Delta}}{2a}}\right) + c$$

Case 3:

$\Delta < 0$

$$I_1(x) = \int \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{-\Delta}}\arctan\left(\frac{x + \frac{b}{2a}}{\frac{\sqrt{-\Delta}}{2a}}\right) + c$$

Example 4.8

Evaluate the integral:

$$I = \int \frac{dx}{x^2 - 6x + 9}$$

Solution:

$$\Delta = (-6)^2 - 4(1)(9) = 36 - 36 = 0$$

$$I = \int \frac{dx}{x^2 - 6x + 9} = \int \frac{dx}{(x-3)^2}$$

Let $t = x - 3$ then $dt = dx$

$$I = \int \frac{dt}{t^2} = \int t^{-2}dt = \frac{-1}{t} + c$$

$$I = \int \frac{dx}{x^2 - 6x + 9} = \frac{-1}{x - 3} + c$$

Example 4.9

Evaluate the integral:

$$I = \int \frac{dx}{x^2 + 2x - 3}$$

Solution:

Method 1: Using the substitution technique.

$$\Delta = (2)^2 - 4(1)(-3) = 16 > 0$$

$$\frac{1}{x^2 + 2x - 3} = \frac{1}{(x+1)^2 - 3 - 1} = \frac{1}{(x+1)^2 - 4} = \frac{1}{(x+1)^2 - 2^2}$$

$$I = \int \frac{dx}{(x+1)^2 - 2^2}$$

Consider $t = x + 1$ then $dt = dx$

$$I = \int \frac{1}{t^2 - 2^2}$$

Remember the elementary primitive

$\frac{1}{t^2 - a^2}$	$\frac{1}{2a} \ln \left\lvert \frac{t-a}{t+a} \right\rvert + c$

$$I = \int \frac{1}{t^2 - 2^2} = \frac{1}{2a} \ln \left| \frac{t-a}{t+a} \right| + c = \frac{1}{2 \times 2} \ln \left| \frac{(x+1)-2}{(x+1)+2} \right| + c = \frac{1}{4} \ln \left| \frac{x-1}{x+3} \right| + c$$

Method 2: Using the partial functions decomposition.

$$I = \int \frac{dx}{x^2 + 2x - 3}$$

$$\Delta = (2)^2 - 4(1)(-3) = 16 > 0$$

$$x_1 = \frac{-2+4}{2} = 1 \qquad x_2 = \frac{-2-4}{2} = -3$$

$$I = \int \frac{dx}{(x-1)(x+3)}$$

$$\frac{1}{(x-1)(x+3)} = \frac{a}{x-1} + \frac{b}{x+3} \text{ .....(1)}$$

Multiply both sides of equation (1) by $(x-1)$ and reducing

$$\frac{1}{(x+3)} = a + \frac{b(x-1)}{x+3}$$

$$\Longrightarrow \ a = \frac{1}{4}$$

$$x \to 1$$

Multiply both sides of equation (1) by $(x+3)$ and reducing

$$\frac{1}{(x-1)} = \frac{a(x+3)}{x-1} + b$$

$$\implies b = \frac{-1}{4}$$

$$x \to -3$$

Hence,

$$I = \int \frac{dx}{(x-1)(x+3)}dx = \int \frac{\frac{1}{4}}{x-1}dx + \int \frac{\frac{-1}{4}}{x+3}dx$$

$$\frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+3}$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + c$$

$$= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + c$$

Example 4.10

Evaluate the integral:

$$I = \frac{dx}{x^2 - 2x + 5}$$

Solution:

$$I = \int \frac{dx}{x^2-2x+5} = \int \frac{dx}{(x-1)^2+5-1} = \int \frac{dx}{(x-1)^2+2^2}$$

In this case, we should use following as substitutions

$$t = x - 1$$

Thus $dt = dx$

Doing the substitutions and evaluating the integral gives:

$$I = \frac{dt}{t^2+2^2}$$

Remember the elementary primitive

$\dfrac{1}{t^2+a^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{t}{a}\right)+c$

$$I=\frac{1}{2}\arctan\left(\frac{t}{2}\right)+c$$

Therefore

$$I=\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)+c$$

Calculation of the Integral:

$$I_2(x)=\int\frac{Ax+B}{ax^2+bx+c}\qquad(\Delta<0,\ a\neq 0)$$

We have to show the numerator $(Ax+B)$ in terms of the derivative of the denominator (ax^2+bx+c)

$$(ax^2+bx+c)'=2ax+b$$

$$Ax+B=\frac{A}{2a}.(2ax+b)+B-\frac{Ab}{2a}$$

$$I_2(x)=\int\frac{Ax+B}{ax^2+bx+c}dx=\int\frac{\frac{A}{2a}.(2ax+b)+B-\frac{Ab}{2a}}{ax^2+bx+c}dx$$

$$=\frac{A}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+\left(B-\frac{Ab}{2a}\right)\int\frac{dx}{ax^2+bx+c}$$

$$\text{So,}\quad I_2(x)=\frac{A}{2a}\ln|ax^2+bx+c|+\left(B-\frac{Ab}{2a}\right)I_1(x)$$

Example 4.11

Evaluate the integral:

$$I=\int\frac{2x+1}{x^2-2x+5}dx$$

Solution:

$\Delta=4-20=-16<0$

$$(x^2-2x+5)'=2x-2$$

$$2x+1=1.(2x-2)+1+2=(2x-2)+3$$

$$I = \int \frac{2x+1}{x^2-2x+5}dx = \int \frac{2x-2}{x^2-2x+5}dx + 3\int \frac{dx}{x^2-2x+5}$$

$$= \ln|x^2-2x+5| + 3I_1(x)$$

$$I = \int \frac{2x+1}{x^2-2x+5}dx = \ln|x^2-2x+5| + \frac{3}{2}\arctan\left(\frac{x-1}{2}\right) + c$$

Example 4.12

Evaluate the integral:

$$I = \int \frac{4x+4}{x^2+x+1}dx$$

Solution:

$\Delta = 1 - 4 = -3 < 0$

$$(x^2+x+1)' = 2x+1$$

We write the numerator in terms of the derivative of the denominator

$$4x+4 = 2(2x+1) + 4 - 2 = 2(2x+1) + 2$$

$$I = \int \frac{4x+4}{x^2+x+1}dx = \int \frac{2(2x+1)+2}{x^2+x+1}dx$$

$$= 2\int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{dx}{x^2+x+1}$$

$$= 2\ln|x^2+x+1| + 2J(x)$$

$$J(x) = \int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4}} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{2}{\sqrt{3}}\arctan\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$$

$$I = 2\ln|x^2+x+1| + \frac{4}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Notice:

The primary objective of the following section is to show that rational functions can be integrated by **breaking them into simpler parts or using the partial functions decomposition.**

Calculation of the Integral:

$$I_3(x) = \frac{\mathcal{P}(x)}{\mathcal{Q}(x)}dx$$

Where $\mathcal{P}(x)$ and $\mathcal{Q}(x)$ are polynomials with $deg(\mathcal{P}) \geq deg(\mathcal{Q})$.

The fraction where the degree of the numerator $\mathcal{P}(x)$ is at least as large as the degree of the denominator $\mathcal{Q}(x)$.

In such cases. The Euclidean division is used in order to reduce the fraction to the sum of a polynomial and a rational fraction like the above integrand in $I_2(x)$ and $I_3(x)$.

Using the Euclidean division of $\mathcal{P}(x)$ by $\mathcal{Q}(x)$, one gets:

$$\mathcal{P}(x) = \mathcal{A}(x)\mathcal{Q}(x) + \mathcal{R}(x)$$

Such that $\mathcal{A}(x)$ is quotient and $\mathcal{R}(x)$ is a remainder

Dividing both side of the equation above by $\mathcal{Q}(x)$, one obtains:

$$\frac{\mathcal{P}(x)}{\mathcal{Q}(x)} = \mathcal{A}(x) + \frac{\mathcal{R}(x)}{\mathcal{Q}(x)}$$

$$I_3(x) = \int \frac{\mathcal{P}(x)}{\mathcal{Q}(x)}dx = \int \left(\mathcal{A}(x) + \frac{\mathcal{R}(x)}{\mathcal{Q}(x)}\right)dx$$

$deg\mathcal{R}(x) < deg\mathcal{Q}(x)$

Therefore

$$I_3(x) = \int \mathcal{A}(x)dx + \int \frac{\mathcal{R}(x)}{\mathcal{Q}(x)}dx$$

We obtain the sum of the integral of a polynomial and the integral of fraction whose numerator is a polynomial with a degree less than the degree of the polynomial of the denominator.

The fraction $\frac{\mathcal{R}(x)}{\mathcal{Q}(x)}$, depending on its classification of the denominator, can be represented in four different cases. These cases are as follows:

Case 1:

The Denominator $\mathcal{Q}(x)$ has distinct roots (or distinct linear factors)

The integral $\frac{\mathcal{R}(x)}{\mathcal{Q}(x)}$ can be represented by partial fraction in the following way:

$$\frac{\mathcal{R}(x)}{\mathcal{Q}(x)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

The following examples show the steps as to how these classes of integrals are solved.

Example 4.13

Evaluate the integral:

$$\int \frac{2x+3}{x(x-1)(x+2)}dx$$

Solution:

The integral can be represented in simple in the following way:

$$\frac{2x+3}{x(x-1)(x+2)} = \frac{a}{x} + \frac{b}{x-1} + \frac{c}{x+2} \text{.....(1)}$$

- Multiply both sides of equation (1) by x and reducing. Then replace x with $x = 0$, one gets $a = \frac{-3}{4}$.
- Multiply both sides of equation (1) by $(x-1)$ and reducing. Then replace x with $x = 1$, one gets $b = \frac{5}{3}$.
- Multiply both sides of equation (1) by $(x+2)$ and reducing. Then replace x with $x = -2$, one gets $c = \frac{-1}{6}$.

$$\int \frac{2x+3}{x(x-1)(x+2)}dx = \int \Big(\frac{\frac{-3}{4}}{x} + \frac{\frac{5}{3}}{x-1} + \frac{\frac{-1}{6}}{x+2}\Big)dx$$

$$= \frac{-3}{4}\int \frac{dx}{x} + \frac{5}{3}\int \frac{dx}{x-1} + \frac{-1}{6}\int \frac{dx}{x+2}$$

$$= \frac{-3}{4}\ln|x| + \frac{5}{3}\ln|x-1| + \frac{-1}{6}\ln|x+2| + c$$

Example 4.14

Evaluate the integral:

$$\int \frac{x+1}{x^3+x^2-6x}dx$$

Solution:

The polynomial x^3+x^2-6x can be factor into three linear factors

$$x^3+x^2-6x = x(x-2)(x+3)$$

We decompose the integral into simpler parts. We get the following sum:

$$\frac{x+1}{x^3+x^2-6x} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

We follow the same steps of the above example for obtaining:

$$A = \frac{-1}{6}\ ,\ B = \frac{3}{10}\ ,\ C = \frac{-2}{15}$$

$$\int \frac{x+1}{x^3+x^2-6x}dx = \frac{-1}{6}\int \frac{dx}{x} + \frac{3}{10}\int \frac{dx}{x-2} + \frac{-2}{15}\int \frac{dx}{x+3}$$

$$= \frac{-1}{6}\ln|x| + \frac{3}{10}\ln|x-2| + \frac{-2}{15}\ln|x+3| + c$$

Example 4.15

Evaluate the integral:

$$\int \frac{x^3+2}{x^2-x-6}dx$$

Solution:

In this case in the integral the degree of the numerator is greater than the degree of the denominator.

We reduce the rational using the Euclidean division to the sum of a polynomial and a rational fraction $\frac{\mathcal{R}(x)}{\mathcal{Q}(x)}$ $(deg\mathcal{R}(x) < deg\mathcal{Q}(x))$.

$$\frac{x^3+2}{x^2-x-6} = \frac{7x+8}{x^2-x-6}$$

To integrate the second term we proceed with the following steps:

1. We factor the denominator $x^2 - x - 6$ into $(x-3)(x+2)$.

2. Write the linear factors in the fraction $\dfrac{7x+8}{x^2-x-6}$, the integrand can be represented in the following way:

$$\frac{7x+8}{x^2-x-6} = \frac{7x+8}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} \text{.....(2)}$$

3. Solve (2) for the constants$A$ and B.

   Multiply both sides of equation (2) by $(x-3)$ and reducing. The replace x with $x = 3$, one gets $a = \dfrac{29}{5}$.

   Multiply both sides of equation (1) by $(x+2)$ and reducing. The replace x with $x = -2$, one gets $b = \dfrac{6}{5}$.

4. Rewrite the integral in its equivalent fraction from by substituting the constants with their specific values.

$$\int \frac{x^3+2}{x^2-x-6}dx = \int \left((x+1) + \frac{7x+8}{x^2-x-6}\right)dx$$

$$= \int (x+1)dx + \int \left(\frac{\frac{29}{5}}{x-3} + \frac{\frac{6}{5}}{x+2}\right)dx$$

$$= \frac{x^2}{2} + x + \frac{29}{5}\int \frac{dx}{x-3} + \frac{6}{5}\int \frac{dx}{x+2}$$

$$= \frac{x^2}{2} + x + \frac{29}{5}\ln|x-3| + \frac{6}{5}\ln|x+2| + c$$

5. Check the answer by differentiating the solution. The result should match the integral.

Case 2:

The Denominator $\mathcal{Q}(x)$ has repeated roots (or repeated Linear Factors)

The integral $\dfrac{\mathcal{R}(x)}{\mathcal{Q}(x)}$ can be represented in the following way:

$$\frac{\mathcal{R}(x)}{\mathcal{Q}(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(cx+d)^2} + ... + \frac{A_3}{(ex+f)^n}$$

The following examples show the steps as to how this class of integrals are solved.

Example 4.16

Evaluate the integral:

$$\int \frac{-x+5}{(x+1)(x-2)^2}dx$$

Solution:

The integral can be decomposed as follows:

$$\frac{-x+5}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

The same steps as the above examples to show $A = \frac{2}{3}$, $C = 1$

To find B, we have to chose an arbitrary value for $x \neq 2$. For example for $x = 0$ we get $B = \frac{-2}{3}$

$$\int \frac{-x+5}{(x+1)(x-2)^2}dx = \int \Big(\frac{\frac{2}{3}}{x+1} + \frac{\frac{-2}{3}}{x-2} + \frac{1}{(x-2)^2}\Big)dx$$

$$= \frac{2}{3}\ln|x+1| - \frac{2}{3}\ln|x-2| - \frac{1}{x-2} + c$$

Example 4.17

Evaluate the integral:

$$\int \frac{5x-1}{(x^2-1)(x+2)^2}dx$$

Solution:

$$\frac{5x-1}{(x^2-1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

$A = \frac{2}{9}$, $B = 3$, $C = 4$, $D = \frac{-11}{3}$

$$\int \frac{5x-1}{(x^2-1)(x+2)^2}dx = \frac{2}{9}\ln|x-1| + 3\ln|x+1| + 4\ln|x+2| - \frac{11}{3(x-2)} + c$$

Integration of Trigonometric Functions

Calculation of the Integral:

$$I(x) = \int \mathcal{F}(\sin x, \cos x)dx$$

Where the integral $\mathcal{F}(\sin x, \cos x)dx$ is a rational function whose numerator and denominator are given by cosine and sine functions. Using the substitution method the integral can be replaced by rational function as studied above.

Case 1:

$$\mathcal{F}(-\sin x, \cos x) = -\mathcal{F}(\sin x, \cos x)$$

We should use the substitution $t = \cos x$ gives $dt = -\sin x dx$

Example 4.18

Evaluate the integral:

$$\int \frac{1}{\sin x}dx$$

Solution:

Let $\mathcal{F}(\sin x, \cos x) = \dfrac{1}{\sin x}$

$$\mathcal{F}(-\sin x, \cos x) = \frac{1}{-\sin x} = -\mathcal{F}(\sin x, \cos x)$$

$$t = \cos x \implies dt = -\sin x dx$$

$$\implies dx = -\frac{dt}{\sin x}$$

$$\int \frac{1}{\sin x}dx = \int \frac{1}{\sin x}\left(-\frac{dt}{\sin x}\right) = -\int \frac{1}{\sin^2 x}dt = -\int \frac{1}{1-\cos^2 x}dt$$

Our substitution $t = \cos x$ gives

$$-\int \frac{1}{1-\cos^2 x}dx = -\int \frac{1}{1-t^2}dt = \int \frac{dt}{t^2-1}$$

Let the following decomposition

$$\frac{1}{t^2-1}=\frac{1}{(t+1)(t-1)}=\frac{-\frac{1}{2}}{t+1}+\frac{\frac{1}{2}}{t-1}$$

$$\int\frac{dt}{t^2-1}=-\frac{1}{2}\int\frac{1}{t+1}+\frac{1}{2}\int\frac{1}{t-1}$$

$$=-\frac{1}{2}\ln|t+1|+\frac{1}{2}\ln|t-1|=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c$$

$$\int\frac{1}{\sin x}dx=\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+c$$

Case 2:

$$\mathcal{F}(\sin x,-\cos x)=-\mathcal{F}(\sin x,\cos x)$$

We should use the substitution $t=\sin x$ gives $dt=-\cos x dx$

Example 4.19

Evaluate the integral:

$$\int\sin^2 x\cos^3 x dx$$

Solution:

Let $\mathcal{F}(\sin x,\cos x)=\sin^2 x\cos^3 x$

$$\mathcal{F}(\sin x,-\cos x)=\sin^2 x(-\cos)^3 x=-\sin^2 x\cos^3 x=-\mathcal{F}(\sin x,\cos x)$$

$$t=\sin x\implies dt=\cos x dx$$

$$\int\sin^2 x\cos^3 x dx=\int t^2\cos^2(\cos x dx)x$$

$$=\int t^2(1-t^2)dt=\int(t^2-1^4)dt=\frac{t^3}{3}-\frac{t^5}{5}+c=\frac{\sin^3 x}{3}-\frac{\sin^5 x}{5}+c$$

Case 3:

$$\mathcal{F}(-\sin x, \cos x) = \mathcal{F}(\sin x, \cos x)$$

We should use the substitution as following:

1. Suppose $t = \tan x$.
2. Then $x = \arctan t$ that implies $dx = \dfrac{dt}{1+t^2}$.
3. Consider $\begin{cases} \cos^2 x = \dfrac{1}{1+t^2} \\ \sin^2 x = \dfrac{t^2}{1+t^2} \end{cases}$

Example 4.20

Evaluate the integral:

$$\int \frac{1}{\cos^2 x} dx$$

Solution:

$$\mathcal{F}(-\sin x, -\cos x) = \frac{1}{(-\cos x)^2} = \frac{1}{\cos^2(x)} = \mathcal{F}(\sin x, \cos x)$$

$t = \tan x$ then $x = \arctan t$ implies $dx = \dfrac{dt}{1+t^2}$

We have $\cos^2 x = \dfrac{1}{1+t^2}$

$$\int \frac{1}{\cos^2(x)} dx = \int \frac{1}{\frac{1}{1+t^2}} \left(\frac{dt}{1+t^2} \right) = \int dt = t + c = \tan x + c$$

Case 4:

If the above cases do not hold.

We should use the substitution as following:

1. Suppose $t = \tan\left(\frac{x}{2}\right)$
2. Then $\frac{x}{2} = \arctan t$ that implies $dx = \dfrac{2dt}{1+t^2}$
3. Consider $\begin{cases} \cos^2 x = \dfrac{1-t^2}{1+t^2} \\ \sin^2 x = \dfrac{2t}{1+t^2} \end{cases}$

Example 4.21

Evaluate the integral:

$$\int \frac{1}{2+\cos x} dx$$

Solution:

$$\mathcal{F}(\sin x, \cos x) = \frac{1}{2+\cos x}$$

We notice $\dfrac{1}{2+\cos x} \neq \dfrac{1}{2-\cos x}$

Let $t = \tan\left(\frac{x}{2}\right)$ then $\frac{x}{2} = \arctan t$, that implies $dx = \dfrac{2dt}{1+t^2}$

$$\int \frac{1}{2+\cos x} dx = \int \frac{1}{2+\dfrac{1-t^2}{1+t^2}} \left(\frac{2dt}{1+t^2}\right) = \int \frac{1+t^2}{t^2+3} \left(\frac{dt}{1+t^2}\right)$$

$$= 2\int \frac{dt}{t^2+3} = 2\int \frac{dt}{t^2+(\sqrt{3})^2} = \frac{2}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \arctan\left(\frac{\tan\left(\frac{x}{2}\right)}{\sqrt{3}}\right) + c$$

4.3 Definite Integrals

You should distinguish carefully between **definite** and **indefinite integrals**. A definite $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function (or family of functions).

If f is continuous on $[a, b]$, then

$$\int_a^b f(x) = F(x)\Big|_a^b = F(b) - F(a)$$

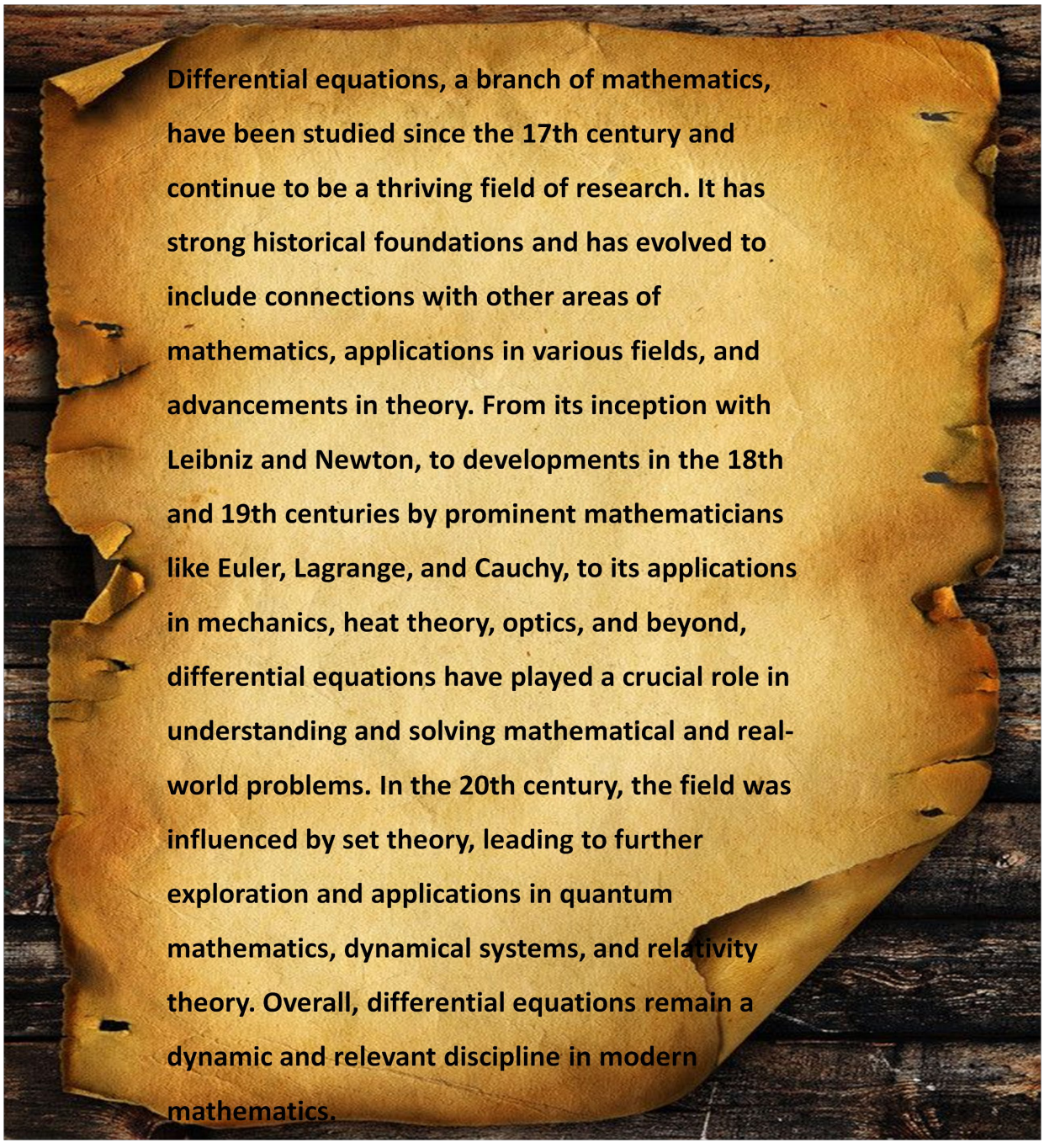
Example 4.22

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t-1}}{t^2}dt$$

$$\int_1^9 \frac{2t^2 - 1}{t^2}dt = \int_1^9 (2 - t^{-2})dt = \Big[2t - (-2+1)t^{(-2+1)}\Big]_1^9$$

$$\Big[2t + \frac{1}{t}\Big]_1^9 = 18 + 9 - 2 - 1 = 24$$

Brief History of Differential Equations



The History of Differential Equations, 1670–1950
Tom Archibald, Craig Fraser and Ivor Grattan-Guinness; Oberwolfach Rep. no. 4, pp. 2729–2794; 2004.

Chapter 5

Ordinary Differential Equations

5.1 Basic Definitions

Differential equations are a fundamental concept in mathematics that involves the study of functions and their rates of change. These equations describe the relationship between a function and its derivatives, representing how a quantity changes concerning one or more independent variables. They widespread applications in various scientific, engineering, and mathematical fields. Here are some key concepts related to differential equations:

- **Differential Equation:** An equation that involves an unknown variables x, y ($y = f(x)$) and its derivatives $\dot{y}, \ddot{y}, \dddot{y}, ..., y^{(n)}$ given by:

$$F\big(x, \dot{y}, \ddot{y}, \dddot{y}, ..., y^{(n)}\big) = 0 \tag{5.1}$$

  - The dot notation $\dot{y}$ represents the first derivative with respect to be independent variable x.
  - The double-dot notation $\ddot{y}$ represents the second derivative.
  - The triple-dot notation $\dddot{y}$ represents the third derivative.
  - The superscript n denotes the n^{th} derivative.

- **Ordinary Differential Equation (ODE):** Involves derivatives with respect to a single variable.

Example 5.1

$$\dot{y} + 2xy = x^2 \ , \quad \dot{y} + 2\ddot{y} = y + \cos x \ , \quad \dot{y} = \frac{y}{x}$$

Partial Differential Equation (PDE): Involves derivatives with respect to multiple variables.

Example 5.2

$$\frac{\partial u}{\partial x} + 2u = xy \quad , \quad u(x, y)$$

Example 5.3

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + y \quad , \quad z(x, y)$$

Order of a Differential Equation: The highest order of the derivative present in the equation.

Degree of a Differential Equation: The power to which the highest order derivative is raised.

Example 5.4

- The equation $\dot{y}(1+\dot{y}^2) = a\ddot{y}$ is a second-order differential equation of second degree.
- The equation $(\dot{y})^n + a\ddot{y} + \dddot{y} = y^n$ is a differential equation of nth order and nth degree.

5.2 Linear First-Order Differential Equation

A linear first-order differential equation is an ordinary differential equation (ODE) of the form:

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{5.2}$$

Here, y is the dependent variable, x is the independent variable, and $P(x)$ and $Q(x)$ are known functions of x.

The general solution to this type of linear first-order ODE can be found using an integrating factor. The integrating factor $u(x)$ is defined as:

$$u(x) = e^{\int P(x)dx} \tag{5.3}$$

Multiplying both sides of the differential equation by the integrating factor gives:

$$u(x)\frac{dy}{dx} + u(x)P(x)y = u(x)Q(x) \tag{5.4}$$

The left-hand side can be expressed as the derivative of the product $u(x)y$:

$$\frac{d}{dx}(u(x)y) = u(x)\frac{dy}{dx} + u(x)P(x)y \tag{5.5}$$

So, the equation becomes:

$$\frac{d}{dx}(u(x)y) = u(x)Q(x) \tag{5.6}$$

Integrating both sides with respect to x and solving for y gives the general solution:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$u(x) = e^{\int P(x)dx}$$

$$y_G = \frac{1}{u(x)} \int u(x)Q(x)dx + c$$

Where c is the constant of integration.

It's worth noting that the specific form of $P(x)$ and $Q(x)$ will determine the exact steps needed to solve the differential equation. The process outlined above is a general method for solving linear first-order ODEs. but details may vary based on the specific functions involved.

Example 5.5

$$\dot{y} + 2y = x$$

Here, $P(x) = 2$ and $Q(x) = x$. The integrating factor $u(x)$ is given as:

$$u(x) = e^{\int P(x)dx} = e^{\int 2dx} = e^{2x}$$

Multiplying the entire differential equation by $u(x)$ gives:

$$e^{2x}\frac{dy}{dx} + 2xe^{2x}y = e^{2x}x$$

Recognizing that the left-hand side is the derivative of $e^{2x}y$,we can write:

$$\frac{d}{dx}(e^{2x}y) = e^{2x}x$$

Integrating both sides with respect to x gives:

$$e^{2x}y = \int xe^{2x}dx + c$$

Thus

$$y = e^{-2x}\big[c + \int xe^{2x}dx\big] = e^{-2x}\big[c + \frac{1}{4}e^{2x}(2x-1)\big] = ce^{-2x} + \frac{1}{4}(2x-1)$$

Exercises: Prove that solutions of the following equations are given as:

$$\dot{y} - \frac{y}{x} = x \ , \ (y = x(c+x))$$

$$\dot{y} - y = \sin x \ , \ (y = ce^{x} - \frac{1}{2}(\sin x + \cos x))$$

5.3 Linear Second-Order Differential Equation

A linear second-order differential equation is an equation that involves the second derivative of an unknown function (dependent variable) with respect to an independent variable, along with the first derivative and the function itself, all linearly combined. The general form of a linear second-order ordinary differential equation (ODE) is given by:

$$a\ddot{y} + b\dot{y} + cy = Q(x) \tag{5.7}$$

Such that:

- y is the unknown function of the independent variable x.
- a, b and c are constants, and a should not be equal to zero, as that would reduce the order of the equation.
- $Q(x)$ is a function of x representing the forcing function or inhomogeneous term.

Solving linear second-order ODEs often involves finding the complementary solution y_H (homogeneous solution) and the particular solution y_P (inhomogeneous solution) and then combining them to obtain the general solution y_G:

$$y_P = y_H + y_P \tag{5.8}$$

The solutions to a linear second-order differential equation depend on the specific coefficients and whether the equation is homogeneous or inhomogeneous. We are going to outline h=the general approach for finding solutions.

5.3.1 Homogeneous Case

The homogeneous linear second-order differential equation is of the form:

$$a\ddot{y} + b\dot{y} + cy = 0 \tag{5.9}$$

Characteristics Equation

Find the **characteristics equation** by substituting $y = e^{kx}$, into the homogeneous equation. The characteristics equation is given by

$$ak^2 + bk + c = 0 \tag{5.10}$$

Solve (5.10) for k:

Solve the characteristic equation to find the roots k_1 and k_2.

The discriminant of the quadratic equation is given by $\Delta = b^2 - 4ac$.

The discriminant determines the nature of the roots k_1 and k_2 and consequently, the form of the general homogeneous solution.

General Homogeneous Solution

The **general homogeneous solution** is given by:

$$y_H = C_1 y_1 + C_2 y_2$$

Where C_1 and C_2 are arbitrary constants determined by initial or boundary conditions.

- **Distinct Real Roots** ($\Delta > 0$):

$$y_1 = e^{k_1 x} \text{ and } y_2 = e^{k_2 x} \text{ and } k_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

The general homogeneous solution is:

$$y_H = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

- **Repeated Real Roots** ($\Delta = 0$):

$$y_1 = e^{kx} \text{ and } y_2 = xe^{kx} \text{ and } k = \frac{-b}{2a}$$

The general homogeneous solution is:

$$y_H = (C_1 + xC_2)e^{kx}$$

- **Complex conjugate Real Roots** ($\Delta < 0$):

$$k_{1,2} = \frac{-b \pm i\sqrt{\Delta}}{2a} \; ; \; \alpha = \frac{-b}{2a} \text{ and } \beta = \frac{i\sqrt{\Delta}}{2a}$$

$$y_1 = e^{\alpha x}\cos\beta x \text{ and } y_2 = e^{\alpha x}\sin\beta x$$

The general homogeneous solutionis:

$$y_H = e^{\alpha x}(C_1\cos\beta x + xC_2\sin\beta x)$$

5.3.2 In-homogeneous Case

The in-homogeneous linear second-order differential equation is of the form:

$$a\ddot{y} + b\dot{y} + cy = Q(x) \tag{5.11}$$

Find the Homogeneous Solution

Follow the steps for the homogeneous case to find the complementary solution y_H

Partial Solution

Find a partial solution y_P that satisfies the in-homogeneous equation. The partial solution in the in-homogeneous term.Here is case for finding the partial solution based on the type of the function $Q(x)$ or in-homogeneous term:

$$Q(x) = P_n(x)e^{\alpha x} \tag{5.12}$$

$P_n(x)$ is a polynomial of degree n

Case 1: α is not a root of the characteristic equation

The partial solution is given by:

$$y_P = (A_0 + A_1x + ... + A_nx^n)e^{\alpha x} \tag{5.13}$$

Case 2: α is a root of the characteristic equation

The partial solution is given by:

$$y_P = x(A_0 + A_1x + ... + A_nx^n)e^{\alpha x} \tag{5.14}$$

Case 3: α is a repeated root of the characteristic equation

The partial solution is given by:

$$y_P = x^2(A_0 + A_1x + ... + A_nx^n)e^{\alpha x} \tag{5.15}$$

To determine the constants $A_0, A_1, ..., A_n$, we differentiate y_P with respect to the variable x, substitute the result into the second equation, and then equate the expressions.

Example 5.6

$$\ddot{y} - 2\dot{y} + 2y = xe^x$$

1. Find the homogeneous solution

$$k^2 - 2k + 2 = 0 \ , \ \Delta = -4$$

$$k_{1,2} = 1 \pm i$$

$$y_H = e^x(C_1 \cos x + C_2 \sin x)$$

2. Fin the partial solution

$$Q(x) = xe^{1.x}$$

$\alpha = 1$ is not a solution for the characteristic equation

$$y_P = (A_0 + A_1 x)e^x$$

$$\dot{y}_P = A_1 e^x + (A_0 + A_1 x)e^x$$

$$\ddot{y}_P = 2A_1 e^x + (A_0 + A_1 x)e^x$$

We substitute into the equation, resulting in two equal polynomials. Following this, we equate coefficients to match the constants. One finds $A_0 = 0$ and $A_1 = 1$ where $y_P = xe^x$.

$$y_G = y_H + y_P = e^x(x + C_1 \cos x + C_2 \sin x)$$

Exercises:

Prove that the solutions of the following equations are given as:

1. $\ddot{y} + 4\dot{y} + 3y = x \quad , \ (y_G = C_1 e^{-x} + C_2 e^{-3x} + \frac{x}{3} - \frac{4}{9})$

2. $\ddot{y} + 2\dot{y} + y = 2e^{-x} \quad , \ (y_G = C_1 e^{-x} + xC_2 e^{-x} + x^2 e^{-x})$

Multivariable functions

The history of multivariable functions is a journey of mathematical exploration spanning centuries. It began with ancient geometric problems and evolved significantly with the advent of calculus in the 17th century, pioneered by Newton and Leibniz, though initially focused on single-variable functions and the 18th century witnessed pivotal contributions from Euler and Clairaut, who introduced partial derivatives and advanced the understanding of functions of multiple variables. In the 19th century, Gauss and Cauchy delved deeper into multivariable calculus. At the same time, Weierstrass and Riemann formalized concepts like continuity and differentiability for functions of several variables, laying the groundwork for modern analysis. Throughout the 20th century, mathematicians like Lebesgue, Cartan, and Weyl further refined the study of multivariable functions, which found applications in diverse fields including physics and optimization. Today, multivariable functions remain essential in modeling complex phenomena and solving real-world problems across various disciplines, continuing to evolve with contemporary developments in mathematics and its applications.

Chapter 6

Functions of Several Variables

6.1 Multivariable Functions

Multivariable functions, also known as multivariate functions, are mathematical functions that depend on more than one variable. In contrast to functions of a single variable, which take a single input and produce a single output, multivariable functions take multiple inputs and produce multiple outputs.

A general form of a multivariable function is represented as:

$$\begin{array}{rccc} f: & \mathbb{R}^n & \to & \mathbb{R}^m \\ (x_1, x_2, ..., x_n) & \mapsto & (x_1, x_2, ..., x_m) & \end{array}$$

Here, f is a function that takes n real-valued inputs and produces m real-valued outputs. The domain of f is denoted as $\mathbb{R}^n$, which is the n-dimentional Euclidean space, and the range is $\mathbb{R}^m$

Example 6.1

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \to & \mathbb{R}^1 \\ (x, y) & \mapsto & f(x, y) = x^2 + y^2 & \end{array}$$

In this case, the function takes two inputs x and y and produces one output, which is the sum of their squares, Graphically, this function represents a surface in three-dimensional space (see Figure 6.1).

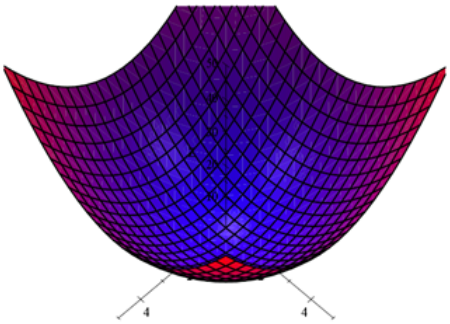


Figure 6.1

6.2 Domain of Multivariable Functions

Example 6.2

Find and sketch the domain for

$$f(x,y) = \frac{e^y}{y - x^2}$$

$D_f = (x,y) \in \mathbb{R}^2 : y \neq x^2$

All point of the plan $\mathbb{R}^2$ except the parabola $y = x^2$ (see Figure 6.2).

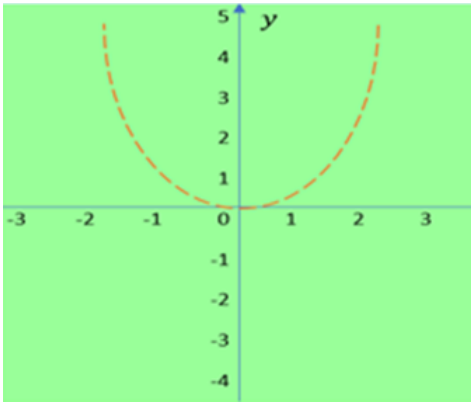


Figure 6.2

Example 6.3

Find and sketch the domain for

$$f(x, y) = y^2 \ln(x - 5)$$

$D_f = \{(x, y) \in \mathbb{R}^2 : y \in \mathbb{R},\ x > 5\}$

The domain is represented as the half-plan $x > 5$ (see Figure 6.3).

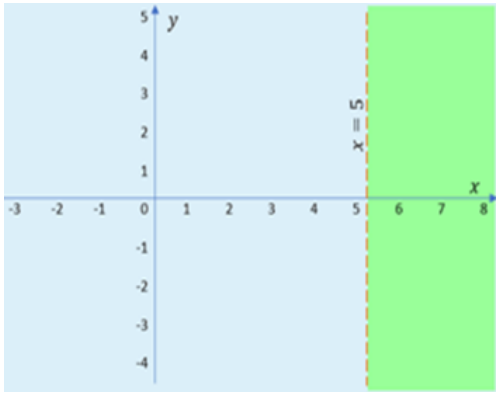


Figure 6.3

6.3 Limit of Multivariable Functions at Point

The concept of limits for multivariable functions is extension of the idea of limits for single-variable functions. In multivariable calculus, we often deal with limits as a function approaches a specific point in multidimensional space.

For a multivariable function $f(x, y)$, the limit as (x, y) approaches a point (a, b) is denoted as: $\lim_{(x,y)\to(a,b)} f(x, y)$.

If the limit exists, it means that as the point (x, y) gets arbitrarily close to (a, b), the values of $f(x, y)$ approach a certain value.

The limit is defined as follows: $\lim_{(x,y)\to(a,b)} f(x, y) = L$.

If for every positive number ϵ, there exists a positive number δ such that if (x, y) is with in δ units of (a, b) (but not equal to (a, b)), then $f(x, y)$ is with in ϵ units of L.

That means as (x, y) gets closer and closer to (a, b), $f(x, y)$ gets closer and closer to the number L.

Here are some examples to illustrate the concept:

Example 6.4

Polynomial Function:

$$f(x, y) = 3x^2 + y$$

$\lim_{(x,y)\to(1,2)} f(x, y) =?$

In this case, you would substitute $x = 1$ and $y = 2$ into the function and simplify. If the result is a finite number, that is the limit. One finds.

$\lim_{(x,y)\to(1,2)} 3x^2 + y = 5$

Example 6.5

Fractional Function:

$$g(x, y) = \frac{x + y}{x - y}$$

$\lim_{(x,y)\to(1,2)} g(x, y) = ?$

Similar to the previous example, substitute $x = 2$ and $y = 1$ into the function and simplify.

We get $\lim_{(x,y)\to(1,2)} \frac{x + y}{x - y} = 3$.

Example 6.6

Piecewise-Defined function :

$$\begin{cases} x^2 + y \quad if \quad x \geq y \\ xy \quad if \quad x > y \end{cases}$$

$\lim\limits_{(x,y)\to(1,2)} h(x,y) = ?$

Determine the limit of h as (x, y) approaches $(1, 1)$ by considering both branches of the piecewise-defined function.

6.4 Continuity of a Function of Two Variables

Definition 6.1

A function f with two variable (x, y) is said to be continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. f is defined at (a, b).
2. $\lim\limits_{(x,y)\to(1,2)} f(x,y) = f(a,b)$.

Example 6.7

Let's consider the function $f(x, y) = \sqrt{x + y}$, and check the continuity at a specific point, say $(a, b) = (1, 4)$.

1. The function $f(x, y) = \sqrt{x + y}$ is defined as long as the expression under the square root is non-negative. So, it's defined at $(1, 4)$.
2. $\lim\limits_{(x,y)\to(1,4)} f(x,y) = \lim\limits_{(x,y)\to(1,4)} \sqrt{x+y} = f(1,4) = \sqrt{5}$.

Since the limit exists and is equal to $\sqrt{5}$, the function $f(x, y) = \sqrt{x + y}$ is continuous at $(1, 4)$.

Example 6.8

$f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$ is continuous on $D_f = \mathbb{R}^2 \setminus \{(0,0)\}$.

f is not continuous at $(0, 0)$ since $(0, 0) \notin D_f$.

6.5 Partial Derivatives

6.5.1 Partial Derivatives of First Order

Partial differentiation is a technique used in calculus to find the rate at which a function changes with respect to one of its several variables, while keeping the other variables constant. The partial derivative of a function with respect to one variable is denoted by a subscript indicating the variable with respect to which the derivative is taken.

Definition 6.2

The **Partial derivatives** of a function with respect to x at the point (a, b) is:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

The Partial derivatives of a function with respect to x at the point (a, b) is:

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

provided these limits exist.

Notations:

Let's consider a function $f(x, y)$ with two variables, x and y. The partial derivatives of f with respect to x and y are denoted by $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ respectively.

- The partial derivative $f_x = \frac{\partial f}{\partial x}$ is found by differentiating f with respect to x while treating y as a constant.
- The partial derivative $f_y = \frac{\partial f}{\partial y}$ is found by differentiating f with respect to y while treating x as a constant.

For a function with more than two variables, the process extends in a similar way. For example, for a function $f(x, y, z)$, the partial derivatives are denoted by $f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$ and $f_z = \frac{\partial f}{\partial z}$, and they are found by differentiating with respect to x, y and z while keeping the other variables constant.

Example 6.9

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x,y) = \sin xy$

$f_x(x,y) = \frac{\partial f(x,y)}{\partial x} = y\cos xy$

$f_y(x,y) = \frac{\partial f(x,y)}{\partial y} = x\cos xy$

Example 6.10

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ where $f(x,y,z) = x^2 + y^2 + z^2$

$f_x(x,y,z) = \frac{\partial f(x,y,z)}{\partial x} = 2x$

$f_y(x,y,z) = \frac{\partial f(x,y,z)}{\partial y} = 2y$

$f_z(x,y,z) = \frac{\partial f(x,y,z)}{\partial z} = 2z$

6.5.2 Partial Derivatives of Second Order

For a function $f(x,y)$ partial derivatives of second order are:

$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$

$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$

$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$

$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2}$

Remark:

f_{xy} and f_{yx} are called mixed partial derivatives.

Example 6.11

$f(x,y) = x^2y + 2xy^2$

$f_{xx} = 2y$

$f_{xy} = 2x + 4y$

$f_{yx} = 2x + 4y$

$f_{xx} = 2x$

6.6 Multiple Integration

A **multiple integral** is an extension of the concept of a single integral to functions of more than one variable .

6.6.1 Double Integral

Double integral are used in physics, engineering and economics to find mass, centre of mass, moment of inertia and probability density functions.

Mastery of double integrals is crucial for excelling in calculus and solving practical problems. A thorough understanding of this concept is necessary for advanced studies and making significant contributions to fields that use calculus.

A double integral is an essential mathematical tool to accurately calculate the volume or area between a surface and a region R in a two-dimensional plane.

It extends the idea of a single integral to integrate a function of one variable in the xy-plane.

The notation for a double integral is $\iint$, and it is typically written as:

$$\iint_R f(x,y)dA$$

Here, R represents the region in the xy-plane over which the integration is performed.

$f(x,y)$ is the **integrand** , and dA represents the **differential area element.**

You often use iterated integrals to compute a double integral, breaking down the process into separate integrals-one for each variable.

The order of integration (choosing whether to integrate with respect to x or y first) can vary based on the problem and ease of computation.

Example 6.12

The iterated from of the double integral might be:

$$\int_c^d \int_a^b f(x,y)dxdy \quad \text{or} \quad \int_a^b \int_c^d f(x,y)dydx$$

Understanding the bounds of integration is crucial, as they define the region over which the integration is performed.

Properties

Consider the rectangle R

1. $\iint_R (f+g)(x,y)dxdy = \iint_R f(x,y)dxdy + \iint_R g(x,y)dxdy$.
2. $\iint_R kf(x,y)dxdy = k\iint_R f(x,y)dxdy$.
3. If $R = R_1 \cup R_2$ (R_1 and R_2 are two rectangles);

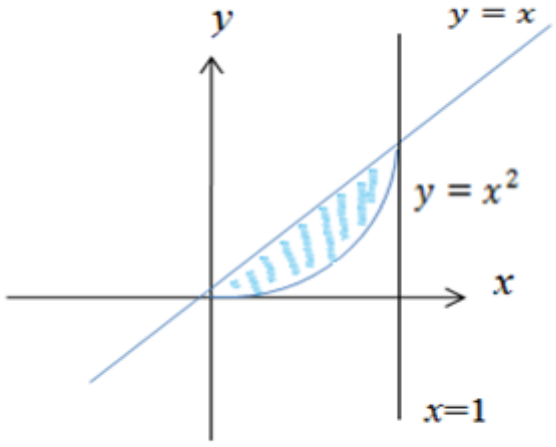
$$\iint_R f(x,y)dxdy = \iint_{R_1} f(x,y)dxdy + \iint_{R_2} f(x,y)dxdy$$

4. If $f(x,y) \leq g(x,y)$, then $\iint_R f(x,y)dxdy \leq \iint_R g(x,y)dxdy$.

Example 6.13

Area calculation

Suppose you want to find the area of the region R bounded by the curves $y = x$ and $y = x^2$ between $x = 0$ and $x = 1$



The double integral for area is:

$$Area = \iint_R 1dA$$

Setting up the double integral and , you get

$$Area = \int_0^1 \int_{x^2}^{x} 1dydx$$

Now, integrate with respect to y first:

$$Area = \int_0^1 [y]_{x^2}^{x} dx$$

Simplify the inner integral

$$Area = \int_0^1 [x - x^2]dx$$

Now, integrate with respect to x

$$Area = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} - 0 + 0$$

Combine terms:

$$Area = \frac{1}{6}$$

Example 6.14

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$R = \{(x, y); \quad 0 \leq x \leq 3 \quad \wedge \quad 0 \leq y \leq 3\}$

$$f(x, y) = \begin{cases} 1 & 0 \leq x \leq 3 \quad 0 \leq y \leq 3 \\ 2 & 0 \leq x \leq 3 \quad 0 \leq y \leq 3 \\ 3 & 0 \leq x \leq 3 \quad 0 \leq y \leq 3 \end{cases}$$

Then;

$$\iint_R f(x,y)dxdy = \int_0^1 \int_0^3 1dxdy + \int_1^2 \int_0^3 2dxdy + \int_2^3 \int_0^3 3dxdy$$

$$\iint_R f(x,y)dxdy = 3 + 12 + 24 = 39$$

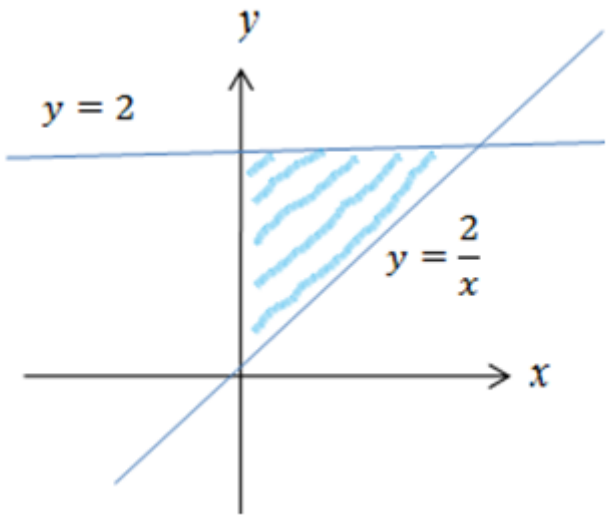
Example 6.15

Calculate the following integral:

$$\iint_S e^{y^2} dxdy$$

$$S = \{(x,y); \quad 0 \leq x \leq 2y, \quad 0 \leq y\}$$

We have $y = \frac{2}{x}$ and $y = 2$



$$\iint_S e^{y^2} dxdy = \int_0^2 \int_0^{2y} e^{y^2} dxdy = \int_0^2 e^{y^2} [x]_0^{2y} dy = \int_0^2 2ye^{y^2} dy = \left[e^{y^2}\right]_0^2 = e^4 - 1$$

6.6.2 Triple Integral

A **triple integral** is an extension of the concept of integration to three dimensions.

It allows us to calculate the volume of a three-dimensional region within a given.

The general from of triple integral is denoted as

$$\iiint_V f(x,y,z)dV$$

Here, V is the region of integration in $3D$ space, $f(x,y,z)$ is the integrand, and dV is the differential volume element. The limits of integration are determined by the region's boundaries.

Triple integrals can be expressed in Cartesian coordinates (x,y,z) as:

$$\iiint_V f(x,y,z)dV = \int_a^b \int_c^d \int_{g(x,y)}^{h(x,y)} f(x,y,z)dzdydx$$

Example 6.16

Calculate the following triple integral:

$$\iiint_V x^2yzdxdydz$$

$$V = \{(x,y,z); \quad 1 \le x \le 2, \quad 0 \le y \le 1, \quad 0 \le z \le 2\}$$

$$\iiint_V x^2yzdxdydz = \int_0^2 \int_0^1 \int_1^2 x^2yzdxdydz = \int_1^2 \int_0^1 \left[\frac{x^3yz}{3}\right]_1^2 dydz = \frac{7}{3}\int_0^2 \int_0^1 yzdydz$$

$$= \frac{7}{3}\int_0^2 [y^2z]_0^1 dz = \frac{7}{3}\int_0^2 \frac{1}{2}zdz = \frac{7}{6}\left[\frac{x^2}{2}\right]_0^2 = \frac{7}{3}$$

Example 6.17

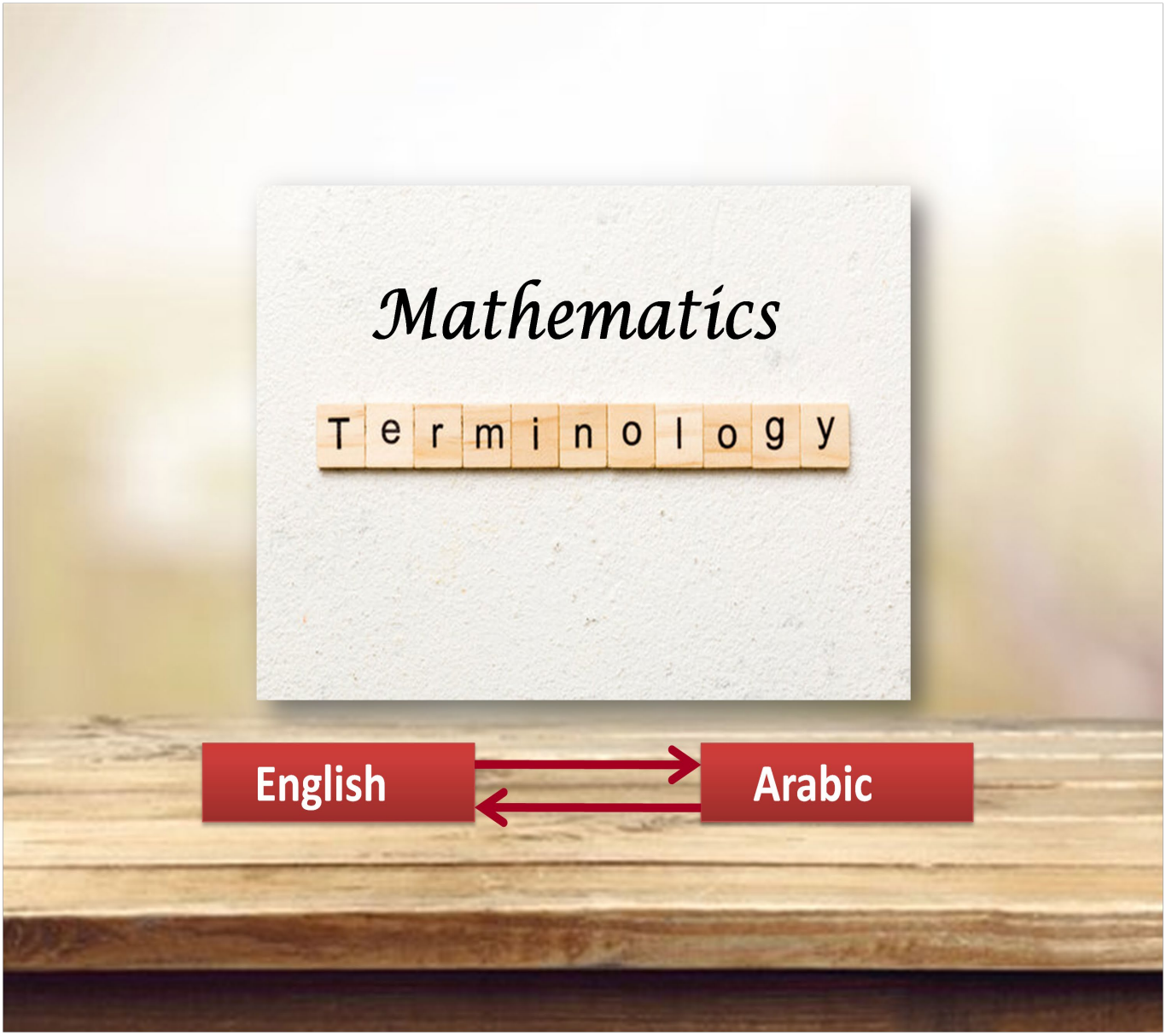
Calculate the following triple integral:

$$\iiint_V 3xy^3z^2dxdydz$$

$V = \{(x, y, z);\ \ -1 \leq x \leq 3, \quad 1 \leq y \leq 4, \quad 0 \leq z \leq 2\}$

$$\iiint_V 3xy^3z^2dxdydz = \int_{-1}^{3}\int_{1}^{4}\int_{0}^{2} 3xy^3z^2dzdydx = \int_{1}^{4}\int_{-1}^{3}\int_{0}^{2} 3xy^3z^2dzdydx$$

$$= \int_{0}^{2}\int_{1}^{4}\int_{-1}^{3} 3xy^3z^2dzdydx = 2040$$


Mathematics
Terminology
English
Arabic

English	Arabic
Vector Space	الفضاء الشعاعي
Linear Maps or linear transformation or Homomorphism	التطبيقات الخطية
Vector	شعاع
Scalars	السلمي
Vector addition	الجمع الشعاعي
Scalar multiplication	الضرب السلمي
Field	الحقل
An internal operation (vector addition)	العملية الداخلية (الجمع الشعاعي)
External operation (scalar multiplication)	العملية الخارجية (الضرب السلمي)
Closed under addition	مستقر بالنسبة للجمع
Closed under scalar multiplication	مستقر بالنسبة للضرب السلمي
Commutativity	التبديل
Associativity	التجميع
Zero vector	الشعاع المعدوم

English	Arabic
Inverse element	الشعاع المعاكس
Distributive	توزيعية
Real vector space	الفضاء الشعاعي الحقيقي
Real coordinate space	فضاء الإحداثيات الحقيقية
Row vector	شعاع أفقي
Column vector	شعاع عمودي
Subspace	فضاء شعاعي جزئي
Linear combination	مزج خطي
Generating set or Spanning set	الجملة المولّدة
Linearly dependent	مرتبطة خطيا
Linearly independent	مستقلة خطيا
Basis	أساس
Standard Basis	الأساس القانوني
Dimension	بُعد

English	Arabic
Image space (or range space)	الصورة المباشرة لتطبيق خطي
Null space (or kernel)	الفضاء المعدوم أو نواة تطبيق خطي
Matrices	المصفوفات
Matrix	مصفوفة
. Determinants	المحددات
An array of numbers	جدول من الأعداد
Arranged	منظمة
Row	سطر
Column	عمود
Linear Algebra	الجبر الخطي
Dynamical systems	جمل الديناميكية
Partial differential equations	معادلات تفاضلية جزئية
Computer graphics	رسومات الحاسوب
Graph theory	نظرية المنحنيات
Identity matrix	مصفوفة الوحدة

English	Arabic
zero matrix	المصفوفة المعدومة
Transpose of matrix	منقول مصفوفة
inverse matrix	مقلوب مصفوفة
Rectangular matrix	المصفوفة المستطيلة
Order (or size)	الدرجة
Row matrix	مصفوفة أفقية
Column matrix	مصفوفة عمودية
Square	مربعة
Main Diagonal (or Principal Diagonal)	القطر الرئيسي
Upper triangular matrix	مصفوفة مثلثية علوية
Lower triangular matrix	مصفوفة مثلثية سفلية
Diagonal matrix	مصفوفة قطرية
Scalar matrix	مصفوفة سلمية

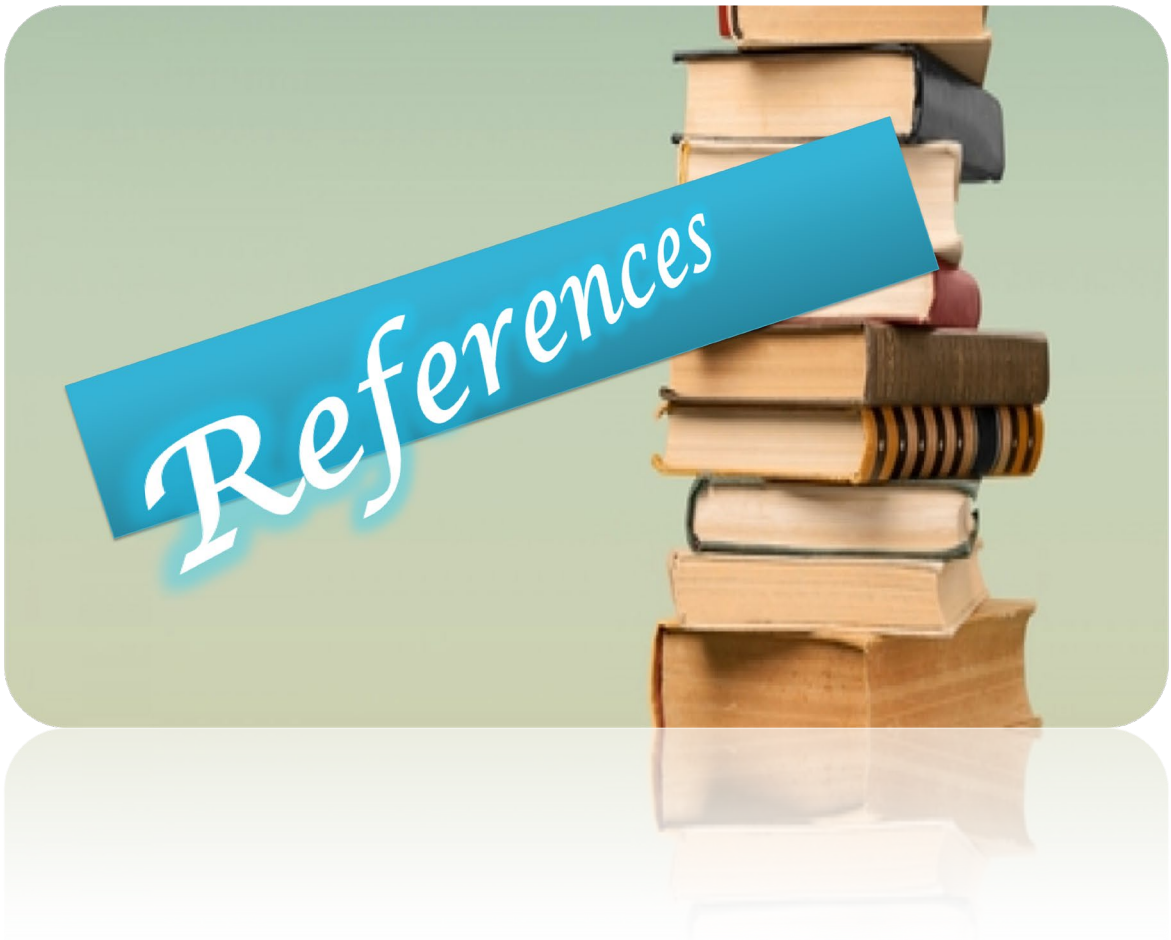
English	Arabic
Vector multiplication	جداء الأشعة
Scalar Product or Inner Product or dot product	الجداء السلمي
Binomials' Identities Formulas	المتطابقات الشهيرة
symmetric matrix	المصفوفة المتناظرة
Skew-Symmetric matrix or anti-symmetric	المصفوفة المتناظرة عكسيا
Trace of square matrix	أثر مصفوفة
inverse of a matrix	مقلوب مصفوفة
Gaussian- Jordan Elimination	حذف غوص-جوردان
Augmented matrix	المصفوفة المُوّسعة
Transition matrix	مصفوفة العبور
Sarrus Rule	قاعدة ساروس
Cofactor Expansion	المعامل المرافق

English	Arabic
Cofactors of a Matrix	المعاملات المرافقة
Minor	المحدد المصغر
Adjoint Matrix	المصفوفة المرافقة
Systems of Linear Equations	جمل المعادلات الخطية
Matrix form	الشكل المصفوفي
Cramer's Rule	قاعدة كرامر
unknown	مجهول
Coefficients matrix	مصفوفة العوامل
Vector of unknowns	شعاع المجاهيل
Vector of constants	شعاع الثوابت
Elementary row operations	العمليات الأولية على الأسطر
Interchange row(or swap)	تبديل الأسطر
Linear combination of rows	المزج الخطي للأسطر
Pivot row	السطر المحوري

English	Arabic
Rank of a matrix	رتبة مصفوفة
Row echelon form	الأسطر المتدرجة
Back-substitution	طريقة التعويض الخلفي
Consistency	تناسق
Free variables	المجاهيل الحرة
Inconsistent	غير متناسق
Primitive	دالة أصلية
Integrations	حساب التكاملآت
Derivation	الاشتقاق
Integration	تكامل
Reverse of a derivative	عكس الاشتقاق
Indefinite integral	التكامل غير المحدود
Partial fraction decompositions	التحليل إلى مجموع كسور الجزئية
Definite integral	التكامل المحدود

English	Arabic
Differential equations	المعادلات التفاضلية
Ordinary Differential equations(ODE)	المعادلات التفاضلية العادية
Partial Differential Equation (PDE)	المعادلة التفاضلية الجزئية
Order of a Differential Equation	الرتبة
Degree of a Differential Equation	الدرجة
Linear first-order differential equation	معادلات تفاضلية خطية من الرتبة الأولى
Linear second-order differential equation	معادلات تفاضلية خطية من الرتبة الثانية
Homogeneous solution	الحل المتجانس
Inhomogeneous	غير متجانس
General solution	الحل العام
Characteristic equation	المعادلة المميزة
Distinct Real Roots	جذور حقيقية مختلفة
Repeated Real Roots	جذور حقيقية مضاعفة
Complex conjugate Roots	جذور مركبة مترافقة

English	Arabic
Partial solution	الحل الخاص
Functions of Several Variables or Multivariable functions	الدوال المتعددة المتغيرات
Domain of the Definition	مجموعة تعريف الدالة
Limit	النهاية
Continuity	الاستمرار
Partial Derivatives	مشتقات جزئية
Multiple Integration	التكامل المتعدد
Double Integral	التكاملات الثنائية
Integrand	الدالة المكاملة
Differential area element	عنصر المساحة التفاضلية
Bounds of integration	حدود التكامل
Triple Integral	التكامل الثلاثي


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